Chapter 4

Vibration Under General Forcing Conditions

(4.1) $F(t) = \frac{F_0}{\pi} + \frac{F_0}{2}\sin\omega t - \frac{2F_0}{\pi}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cos n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_0}{\pi k} + \frac{F_0}{2k}\cdot\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(\omega t - \phi_1)$$

$$- \frac{2F_0}{\pi k}\sum_{n=2,4,6,\ldots}\frac{1}{(n^2-1)}\cdot\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta n r)^2}}\cos(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$, $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right)$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

(4.2) From the solution of problem 1.63,

$$F(t) = \begin{cases} (2F_0 t/\tau) \; ; & 0 \le t \le \tau/2 \\ -(2F_0 t/\tau) + 2F_0 \; ; & \tau/2 \le t \le \tau \end{cases}$$

Fourier series representation of $F(t)$ is

$$F(t) = \frac{F_0}{2} - \frac{4F_0}{\pi^2}\sum_{n=1,3,5,\ldots}^{\infty}\frac{1}{n^2}\cos n\omega t$$

$$\therefore \quad x(t) = \frac{F_0}{2k} - \frac{4F_0}{\pi^2 k}\sum_{n=1,3,\ldots}^{\infty}\frac{1}{n^2}\frac{\cos(n\omega t - \phi_n)}{\sqrt{(1-r^2n^2)^2+(2\zeta n r)^2}}$$

where $r = \frac{\omega}{\omega_n}$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$.

(4.3) $F(t) = \frac{8F_0}{\pi^2}\sum_{n=1,3,5,\ldots}(-1)^{\frac{n-1}{2}}\sin\frac{n\omega t}{n^2}$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{8F_0}{\pi^2 k}\sum_{n=1,3,5,\ldots}\frac{1}{n^2}(-1)^{\frac{n-1}{2}}\frac{1}{\sqrt{(1-n^2r^2)^2+(2\zeta n r)^2}}\sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

(4.4) $F(t) = \frac{F_0}{2} + \frac{F_0}{\pi}\sum_{n=1,2,\ldots}^{\infty}\frac{1}{n}\sin n\omega t$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_o}{2k} + \frac{F_o}{\pi k} \sum_{n=1,2,\ldots}^{\infty} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \sin(n\omega t - \phi_n)$$

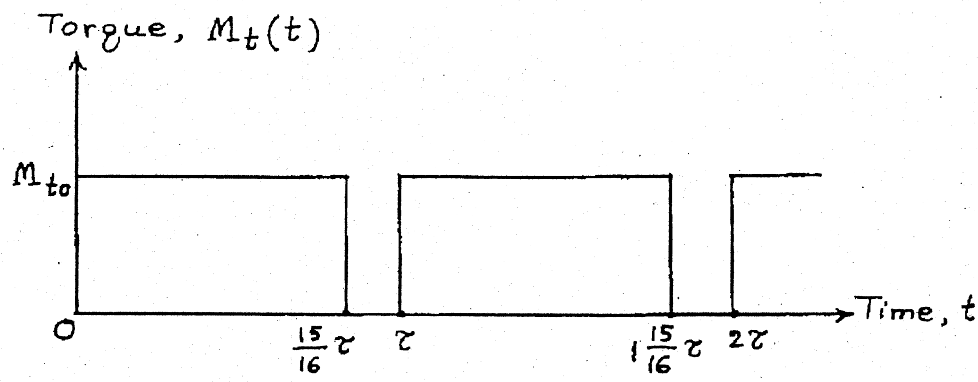
where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.5 From Example 1.12, $F(t) = \frac{F_o}{\pi}\left[\frac{\pi}{2} - \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \sin n\omega t\right]$ where $\omega = \frac{2\pi}{\tau}$

$$x(t) = \frac{F_o}{2k} - \frac{F_o}{\pi k} \sum_{n=1,2,3,\ldots} \frac{1}{n} \cdot \frac{1}{\sqrt{(1-n^2r^2)^2 + (2\zeta n r)^2}} \cdot \sin(n\omega t - \phi_n)$$

where $r = \omega/\omega_n$ and $\phi_n = \tan^{-1}\left(\frac{2\zeta n r}{1-n^2r^2}\right)$

4.6



Torque transmitted to driven gear is shown in the figure. It can be expressed as:

$$M_t(t) = \frac{a_0}{2} + \sum_{n-1}^{\infty} \left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right)$$

$$\text{where} \quad \omega = 2\,\pi\left(\frac{1000}{60}\right) = 104.72 \text{ rad/sec}$$

$$\tau = \frac{2\,\pi}{\omega} = 0.06 \text{ sec} \quad ; \quad \frac{15}{16}\,\tau = 0.05625 \text{ sec}$$

$$M_t(t) = \begin{cases} M_{t0} = 1000 \text{ N--m} \; ; \; 0 \le t \le \frac{15}{16}\,\tau \\ 0 \; ; \; \frac{15}{16}\,\tau \le t \le \tau \end{cases}$$

$$a_0 = \frac{2}{\tau}\int_0^{\tau} M_t(t)\,dt = \frac{2}{0.06}\int_0^{0.05625} (1000)\,dt = \frac{2000}{0.06}(0.05625) = 1875.0 \text{ N--m}$$

$$a_n = \frac{2}{\tau}\int_0^{\tau} M_t(t)\cos n\,\omega\,t\,dt = \frac{2}{\tau} M_{t0} \int_0^{0.05625} \cos n\,\omega\,t\,dt$$

$$= \frac{2\,M_{t0}}{\tau}\left(\frac{\sin n\,\omega\,t}{n\,\omega}\right)_0^{0.05625} = \frac{318.3091}{n} \sin 5.8905\,n \text{ N--m}$$

$$b_n = \frac{2}{\tau} \int_0^{\tau} M_t(t) \sin n \omega t \, dt = \frac{2 M_{t0}}{\tau} \int_0^{0.05625} \sin n \omega t \, dt$$

$$= \frac{2 M_{t0}}{\tau} \left(- \frac{\cos n \omega t}{n \omega} \right)_0^{0.05625} = \frac{318.3091}{n} \left(1 - \cos 5.8905 \, n \right) \text{ N–m}$$

$$k_t = \frac{GJ}{\ell} = G \left(\frac{\pi d^4}{4} \right) \frac{1}{\ell} = (80 \, (10^9)) \left(\frac{\pi \, (0.05)^4}{4} \right) \frac{1}{1} = 392{,}700 \text{ N–m/rad}$$

$$\omega_n = \sqrt{\frac{k_t}{J_0}} = \sqrt{\frac{392700}{0.1}} = 1981.6660 \text{ rad/sec}$$

Equation of motion:

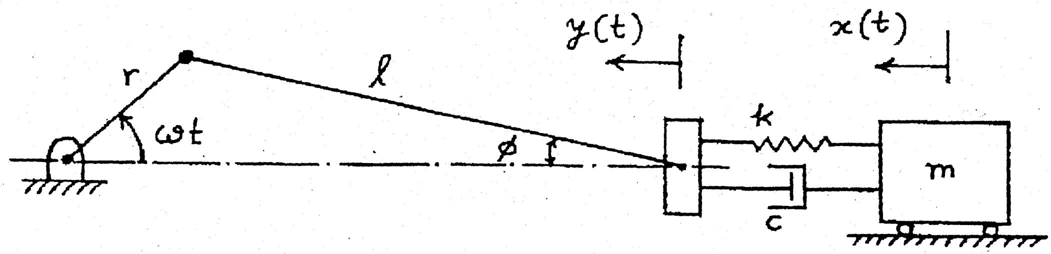
$$J_0 \ddot{\theta} + k_t \theta = M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty} \left(a_n \cos n \omega t + b_n \sin n \omega t \right)$$

Response:

$$\theta(t) = \frac{a_0}{2 k_t} + \sum_{n=1}^{\infty} \left\{ \frac{a_n \cos n \omega t + b_n \sin n \omega t}{k_t - J_0 (n \omega)^2} \right\}$$

$$= 0.0023873 + \sum_{n=1}^{\infty} \left\{ \frac{318.3091 \sin 5.8905 \, n \cos \omega t + 318.3091 \, (1 - \cos 5.8905 \, n) \sin n \omega t}{n \, (392700.0 - 1096.6278 \, n^2)} \right\} \text{ rad}$$

4.7



Base motion is given by:

$$y(t) = r + \ell - r \cos \omega t - \ell \cos \phi = r + \ell - r \cos \omega t - \ell \sqrt{1 - \sin^2 \phi} \quad (1)$$

Using $\ell \sin \phi = r \sin \omega t$, Eq. (1) becomes

$$y(t) = r + \ell - r \cos \omega t - \ell \sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \quad (2)$$

Using the approximation:

$$\sqrt{1 - \frac{r^2}{\ell^2} \sin^2 \omega t} \approx 1 - \frac{r^2}{2 \ell^2} \sin^2 \omega t \quad (3)$$

Eq. (2) can be expressed as

$$y(t) = r + \ell - r \cos \omega t - \ell \left(1 - \frac{1}{2} \frac{r^2}{\ell^2} \sin^2 \omega t \right)$$

$$= r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t \quad (4)$$

Equation of motion:

$$m\ddot{x} + c\dot{x} + kx = ky + c\dot{y}$$

$$= k\,r - k\,r\cos\omega t + \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 - \frac{k\ell}{4}\left(\frac{r}{\ell}\right)^2 \cos 2\omega t + \cdots$$

$$+ c\,r\,\omega\sin\omega t + \frac{c\ell}{4}\left(\frac{r}{\ell}\right)^2 (2\omega)\sin 2\omega t + \cdots \tag{5}$$

Solution of Eq. (5) can be found by adding the solutions due to each term on the right hand side of Eq. (5).

Solution due to constant term, F_0 (terms 1 and 3 on the r.h.s. of Eq. (5)):

$$x(t) = \frac{F_0}{k}\left[1 - \frac{1}{\sqrt{1-\varsigma^2}}\, e^{-\varsigma\omega_n t}\cos(\omega_d t - \phi)\right] \tag{6}$$

$$\text{where}\quad \phi = \tan^{-1}\left(\frac{\varsigma}{\sqrt{1-\varsigma^2}}\right)$$

Solution due to sinusoidal term, $F_0 \sin \Omega t$ (terms 5 and 6 on the r.h.s. of Eq. (5)):

$$x(t) = X\sin(\Omega t - \phi_0) \tag{7}$$

$$\text{where}\quad X = \frac{F_0}{\left[(k - m\,\Omega^2)^2 + c^2\,\Omega^2\right]^{\frac{1}{2}}} \quad\text{and}\quad \phi_0 = \tan^{-1}\left(\frac{c\,\Omega}{k - m\,\Omega^2}\right) \tag{8}$$

Solution due to cosine term, $F_0 \cos \Omega t$ (terms 2 and 4 in Eq. (5)):

$$x(t) = X\cos(\Omega t - \phi_0) \tag{9}$$

$$\text{where}\quad X = \frac{F_0}{\left[(k - m\,\Omega^2)^2 + c^2\,\Omega^2\right]^{\frac{1}{2}}} \quad\text{and}\quad \phi_0 = \tan^{-1}\left(\frac{c\,\Omega}{k - m\,\Omega^2}\right) \tag{10}$$

For given data, $\varsigma = \dfrac{c}{2\sqrt{m\,k}} = \dfrac{10}{2\sqrt{1\,(100)}} = 0.5$, $\dfrac{r}{\ell} = 0.1$, $\omega = 100$, $2\,\omega = 200$, etc. and the solution of Eq. (5) can be obtained by using Eqs. (6) to (8) suitably.

4.8 Base motion can be represented by Fourier series as (from Example 1.12):

$$y(t) = \frac{Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin\omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \tag{1}$$

Equation of motion of mass:

$$m \ddot{x} + c (\dot{x} - \dot{y}) + k (x - y) = 0 \tag{2}$$

Since y(t) is composed of several terms, the solution of Eq. (2) can be found by superposingg the solutions corresponding to each of the terms appearing in Eq. (1). When $y(t) = Y/2$, constant, equation of motion becomes:

$$m \ddot{x} + c \dot{x} + k x = \frac{k Y}{2} = \text{constant} \tag{3}$$

The steady state solution of Eq. (3) is given by (see Example 4.3):

$$x(t) = \frac{Y}{2} \left[1 - \frac{1}{\sqrt{1 - \varsigma^2}} e^{-\varsigma \omega_n t} \operatorname{coss} (\omega_d t - \phi) \right] \tag{4}$$

$$\text{where} \quad \phi = \tan^{-1} \left(\frac{\varsigma}{\sqrt{1 - \varsigma^2}} \right) \tag{5}$$

When $y(t) = A \sin \Omega t$, the steady state solution of Eq. (2) is given by Eq. (3.67):

$$x(t) = A \sin (\Omega t - \phi) \tag{6}$$

$$\text{where} \quad A = \left[\frac{1 + (2 \varsigma r)^2}{(1 - r^2)^2 + (2 \varsigma r)^2} \right]^{\frac{1}{2}} \tag{7}$$

$$\phi = \tan^{-1} \left(\frac{2 \varsigma r^3}{1 + r^2 (4 \varsigma^2 - 1)} \right) \tag{8}$$

$$\text{and} \quad r = \frac{\Omega}{\omega_n}$$

4.9 From solution of Problem 4.7, we can express the base motion as:

$$y(t) = r - r \cos \omega t + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 - \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \cdots \tag{1}$$

Equation of motion:

$$m \ddot{x} + k (x - y) \pm \mu N = 0$$

or

$$m \ddot{x} + k x \pm \mu N = k y$$

$$= \left\{ k r + \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} - k r \cos \omega t - \frac{k \ell}{4} \left(\frac{r}{\ell} \right)^2 \cos 2 \omega t + \cdots \tag{2}$$

For given numerical data, Eq. (2) becomes:

$$\ddot{x} + 100\,x \pm 0.981 = 100\,y$$

$$= \left\{ 100\,(0.1) + \frac{100\,(1)}{4} \left(\frac{0.1}{1} \right)^2 \right\} - (100)\,(0.1)\cos 100\,t - \frac{100\,(1)}{4} \left(\frac{0.1}{1} \right)^2 \cos 200\,t - \cdots$$

$$= 10.25 - 10 \cos 100\,t - 0.25 \cos 200\,t - \cdots \tag{3}$$

Using the definition of equivalent damping constant, c_{eq}, the solution of Eq. (3) can be found by superposition.

$$c_{eq} = \frac{4\,\mu\,N}{\pi\,\Omega\,X} = \frac{4\,\mu\,m\,g}{\pi\,\Omega\,X} \tag{4}$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (3) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \left\{ k\,r + \frac{k\,\ell}{4} \left(\frac{r}{\ell} \right)^2 \right\} = r + \frac{\ell}{4} \left(\frac{r}{\ell} \right)^2 \tag{5}$$

Steady state solution due to harmonic term, $F_0 \cos \Omega\,t$, on the r.h.s. of Eq. (3) (from Eq. (3.93)):

$$x(t) = X \cos (\Omega\,t - \phi) \tag{6}$$

$$\text{where} \quad X = \frac{F_0}{k} \left[\frac{1 - \left(\frac{4\,\mu\,N}{\pi\,F_0} \right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2} \right)^2} \right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1} \left\{ \frac{\pm \frac{4\,\mu\,N}{\pi\,F_0}}{\left[1 - \left(\frac{4\,\mu\,N}{\pi\,F_0} \right)^2 \right]^{\frac{1}{2}}} \right\} \tag{7}$$

$N = m\,g = 1\,(9.81) = 9.81$ Newtons, $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{100}{1}} = 10$ rad/sec, and + sign is to be used in Eq. (7) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (5) and (6) can be superposed suitably to find the complete steady state solution of Eq. (3).

4.10 Base motion can be represented by Fourier series as (see solution of Problem 4.8 or Example 1.12):

$$y(t) = \frac{Y}{\pi} \left\{ \frac{\pi}{2} - \left(\sin \omega\,t + \frac{1}{2} \sin 2\,\omega\,t + \frac{1}{3} \sin 3\,\omega\,t + \cdots \right) \right\} \tag{1}$$

Equation of motion of mass:

$$m\,\ddot{x} + k\,((x - y) \pm \mu\,N = 0$$

or

$$m\ddot{x} + k x \pm \mu N = k y$$

$$= \frac{k Y}{\pi}\left[\frac{\pi}{2} - \left\{\sin \omega t + \frac{1}{2}\sin 2\omega t + \frac{1}{3}\sin 3\omega t + \cdots\right\}\right] \quad (2)$$

Using the definition of equivalent viscous damping constant:

$$c_{eq} = \frac{4\mu N}{\pi \Omega X} = \frac{4\mu m g}{\pi \Omega X} \quad (3)$$

where Ω is the frequency of the harmonic force and X is the amplitude of the mass, the solution of Eq. (2) can be determined using the superposition principle.

Steady state solution due to constant term, F_0, on the r.h.s. of Eq. (2) (from Example 4.3):

$$x(t) = \frac{F_0}{k} = \frac{k Y}{2 k} = \frac{Y}{2} \quad (4)$$

Steady state solution due to harmonic term, $F_0 \sin \Omega t$, on the r.h.s. of Eq. (2) (from Eq. (3.93)):

$$x(t) = X \sin(\Omega t - \phi)$$

$$\text{where} \quad X = \frac{F_0}{k}\left[\frac{1 - \left(\frac{4\mu N}{\pi F_0}\right)^2}{\left(1 - \frac{\Omega^2}{\omega_n^2}\right)^2}\right]^{\frac{1}{2}} \quad \text{and} \quad \phi = \tan^{-1}\left\{\frac{\pm\frac{4\mu N}{\pi F_0}}{\left[1 - \left(\frac{4\mu N}{\pi F_0}\right)^2\right]^{\frac{1}{2}}}\right\} \quad (6)$$

and + sign is to be used in Eq. (6) for $\Omega < \omega_n$ and - sign for $\Omega > \omega_n$. Equations (4) and (5) can be superposed suitably to find the complete steady state solution of Eq. (2).

(4.11) From solution of problem 1.70,

$$F(t) = 9.9584 - 20.1587 \cos 10.472 t + 23.5253 \sin 10.472 t$$
$$+ 3.3099 \cos 20.944 t + 12.2646 \sin 20.944 t$$
$$+ 3.7719 \cos 31.416 t - 0.4064 \sin 31.416 t \quad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{15000}{1}} = 122.4745$ rad/sec ; $\zeta = 0.1$; $r = 0.0855$

The steady state solution is given by Eq. (4.13):

$$x_p(t) = \frac{9.9584}{k} - \frac{20.1587}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\cos(10.472t-\phi_1)$$

$$+ \frac{23.5253}{k}\frac{1}{\sqrt{(1-r^2)^2+(2\zeta r)^2}}\sin(10.472t-\phi_1)$$

$$+ \frac{3.3099}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\cos(20.944t-\phi_2)$$

$$+ \frac{12.2646}{k}\frac{1}{\sqrt{(1-4r^2)^2+(4\zeta r)^2}}\sin(20.944t-\phi_2)$$

$$+ \frac{3.7719}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\cos(31.416t-\phi_3)$$

$$- \frac{0.4064}{k}\frac{1}{\sqrt{(1-9r^2)^2+(6\zeta r)^2}}\sin(31.416t-\phi_3) \qquad (E_2)$$

where $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(0.017226) = 0.0172$ rad

$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(0.035229) = 0.0352$ rad

$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(0.054913) = 0.0549$ rad

Noting that $2\zeta r = 2(0.1)(0.0855) = 0.0171$
and $(1-r^2) = 0.9927$, Eq. (E_2) can be rewritten as

$$x_p(t) = [6.6389 - 13.7821\cos(10.472t - 0.0172)$$
$$+ 15.7965\sin(10.472t - 0.0172) + 2.2715\cos(20.944t - 0.0352)$$
$$+ 8.4168\sin(20.944t - 0.0352) + 2.6876\cos(31.416t - 0.0549)$$
$$- 0.2896\sin(31.416t - 0.0549] \times 10^{-4}\ m.$$

4.12 From problem 1.69,

$$F(t) = 1137.5 - 414.9436\cos 523.6t + 150.3139\sin 523.6t$$
$$+ 28.6058\cos 1047.2t - 146.1706\sin 1047.2t$$
$$+ 35.7278\cos 1570.8t + 55.1546\sin 1570.8t \qquad (E_1)$$

$\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{8000}{0.5}} = 126.4911$ rad/sec ; $\zeta = 0.06$

$r = \frac{\omega}{\omega_n} = \frac{523.6}{126.4911} = 4.1394$; $r^2 = 17.1348$; $1-r^2 = -16.1348$

$2\zeta r = 2(0.06)(4.1394) = 0.4967$

$$x_p(t) = \frac{1137.5}{k} - \frac{414.9436}{k} \frac{\cos(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}}$$
$$+ \frac{150.3139}{k} \frac{\sin(523.6t - \phi_1)}{\sqrt{(1-r^2)^2 + (2\zeta r)^2}} + \frac{28.6058}{k} \frac{\cos(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}}$$
$$- \frac{146.1706}{k} \frac{\sin(1047.2t - \phi_2)}{\sqrt{(1-4r^2)^2 + (4\zeta r)^2}} + \frac{35.7278}{k} \frac{\cos(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}}$$
$$+ \frac{55.1546}{k} \frac{\sin(1570.8t - \phi_3)}{\sqrt{(1-9r^2)^2 + (6\zeta r)^2}} \qquad (E_2)$$

Since $\phi_1 = \tan^{-1}\left(\frac{2\zeta r}{1-r^2}\right) = \tan^{-1}(-0.0308) = -0.0308$

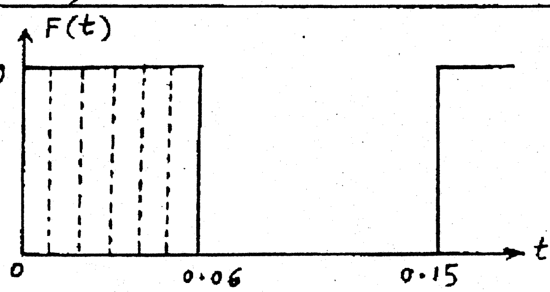
$$\phi_2 = \tan^{-1}\left(\frac{4\zeta r}{1-4r^2}\right) = \tan^{-1}(-0.0147) = -0.0147$$

$$\phi_3 = \tan^{-1}\left(\frac{6\zeta r}{1-9r^2}\right) = \tan^{-1}(-0.00973) = -0.00973$$

the steady-state solution, Eq. (E_2), can be expressed as

$$x_p(t) = 0.1422 - 0.003211 \cos(523.6t + 0.0308)$$
$$+ 0.001163 \sin(523.6t + 0.0308) + 5.2921 \times 10^{-5} \cos(1047.2t + 0.0147)$$
$$- 2.7042 \times 10^{-4} \sin(1047.2t + 0.0147) + 2.9145 \times 10^{-5} \cos(1570.8t + 0.00973)$$
$$+ 4.4992 \times 10^{-5} \sin(1570.8t + 0.00973) \quad m$$

4.13 $\omega_n = \sqrt{\frac{5 \times 10^6}{10 \times 10^3}} = 22.3607 \ \frac{rad}{sec}$



$\tau = 0.15$ sec

$\omega = \frac{2\pi}{\tau} = 41.888$ rad/sec

$r = \frac{\omega}{\omega_n} = 1.8733$, $r^2 = 3.5093$

$\zeta = 0$

Following data is used in Program # 1 to find Fourier coefficients in the expansion of $F(t)$:

t, sec	0.01	0.02	0.03	0.04	0.05	0.06	0.07	...	0.15
F(t), kN	400	400	400	400	400	400	0	...	0

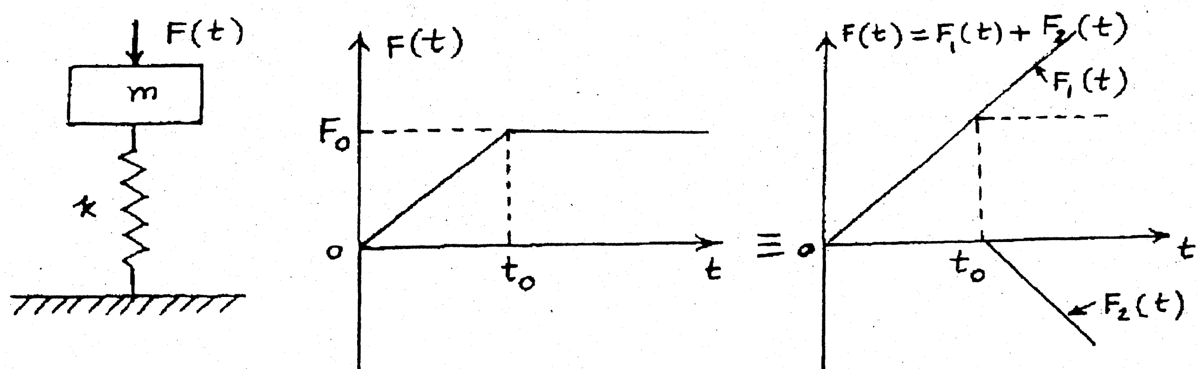
Result is:

$$F(t) = 160.0 + 25.5002 \cos 41.888t + 242.6276 \sin 41.888t$$
$$- 75.3884 \cos 83.776t + 16.0237 \sin 83.776t$$
$$+ 16.4806 \cos 125.664t + 50.7237 \sin 125.664t$$
$$- 62.3538 \cos 167.552t + 27.7604 \sin 167.552t$$
$$+ \cdots \quad kN$$

Since $\zeta = 0$ and all $\phi_j = 0$, $j = 1, 2, \cdots$, the steady-state response of the water tank, Eq. (4.13), becomes

$$x_p(t) = 0.032 + 2.0325 \times 10^{-3} \cos 41.888t$$
$$+ 19.3383 \times 10^{-3} \sin 41.888t - 1.1566 \times 10^{-3} \cos 83.776t$$
$$+ 0.2458 \times 10^{-3} \sin 83.776t + 0.1078 \times 10^{-3} \cos 125.664t$$
$$+ 0.3317 \times 10^{-3} \sin 125.664t - 0.2334 \times 10^{-3} \cos 167.552t$$
$$+ 0.1007 \times 10^{-3} \sin 167.552t + \cdots \quad m$$

4.14



Forcing function can be considered as the sum of two ramp functions, $F_1(t) = \frac{F_0\, t}{t_0}$ and $F_2(t) = -\frac{F_0 (t - t_0)}{t_0}$.

Response of the casting (undamped spring-mass system) to F_1 is given by

$$x_1(t) = \frac{F_0}{k} \left(\frac{t}{t_0} - \frac{\sin \omega_n t}{\omega_n t_0} \right) \quad \text{for } t \geq 0 \qquad (E_1)$$

Response due to F_2 can be obtained from Eq. (E_1) by replacing t by $t - t_0$ and F_0 by $-F_0$:

$$x_2(t) = -\frac{F_0}{k} \left\{ \frac{t - t_0}{t_0} - \frac{\sin \omega_n (t - t_0)}{\omega_n t_0} \right\} \quad \text{for } t \geq t_0 \qquad (E_2)$$

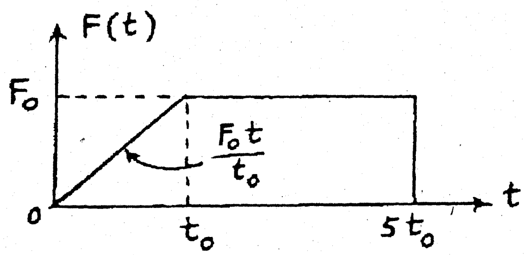
Total response of the casting is given by

$$x(t) = x_1(t) + x_2(t) = \frac{F_o}{k}\left\{1 + \frac{\sin\omega_n(t-t_o) - \sin\omega_n t}{\omega_n t_o}\right\}$$

for $t \geq t_o$ $\quad (E_3)$

(4.15)

$$F(t) = \begin{cases} (F_o t/t_o) & ; \; 0 \le t \le t_o \\ F_o & ; \; t_o \le t \le 5t_o \\ 0 & ; \; t > 5t_o \end{cases} \quad \text{---}(E_1)$$



Response of the anvil is given by [Eq. (4.33) for an undamped system]:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau \qquad (E_2)$$

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{m\omega_n t_o}\int_0^t \tau\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_o}{k\,t_o}\left(t - \frac{1}{\omega_n}\sin\omega_n t\right) \qquad (E_3)$$

For $t_o \le t \le 5t_o$:

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_o}{m\omega_n t_o}\int_0^{t_o}\tau\sin\omega_n(t-\tau)\,d\tau + \frac{F_o}{m\omega_n}\int_{t_o}^{t}\sin\omega_n(t-\tau)\,d\tau$$

$$= \frac{F_o}{m\omega_n t_o}\Big[\sin\omega_n t\left\{\frac{1}{\omega_n^2}\cos\omega_n t_o + \frac{t_o}{\omega_n}\sin\omega_n t_o - \frac{1}{\omega_n^2}\right\}$$

$$- \cos\omega_n t\left\{\frac{1}{\omega_n^2}\sin\omega_n t_o - \frac{t_o}{\omega_n}\cos\omega_n t_o\right\}\Big]$$

$$+ \frac{F_o}{m\omega_n}\left\{\frac{1}{\omega_n}\cos(\omega_n t - \omega_n\tau)\right\}_{t_o}^{t}$$

$$= \frac{F_o}{k\,\omega_n t_o}\left[\sin\omega_n(t-t_o) - \sin\omega_n t + \omega_n t_o\right] \qquad (E_4)$$

For $t > 5t_0$:

$$x(t) = \frac{1}{m\omega_n}\left[\int_0^{t_0} \frac{F_0 \tau}{t_0} \sin \omega_n (t-\tau)\, d\tau + F_0 \int_{t_0}^{5t_0} \sin \omega_n (t-\tau)\, d\tau\right]$$

$$= \frac{F_0}{m\omega_n^2 t_0}\left[\frac{1}{\omega_n} \sin \omega_n (t-t_0) + t_0 \cos \omega_n (t-t_0) - \frac{1}{\omega_n} \sin \omega_n t\right]$$

$$+ \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n} \cos \omega_n (t-\tau)\right]_{t_0}^{5t_0}$$

$$= \frac{F_0}{k\omega_n t_0}\left[\sin \omega_n (t-t_0) - \sin \omega_n t + t_0 \omega_n \cos \omega_n (t-5t_0)\right] \quad (E_5)$$

4.16 From Eq.(4.33), $x(t) = \frac{1}{m\omega_d} \int_0^t F_0\, e^{-\alpha\tau}\, e^{-\gamma\omega_n(t-\tau)} \sin \omega_d (t-\tau)\, d\tau$

$$= \frac{F_0}{m\omega_d} e^{-\gamma\omega_n t} \int_0^t e^{-(\alpha-\gamma\omega_n)\tau} \sin \omega_d (t-\tau)\, d\tau$$

$$x(t) = \frac{F_0\, e^{-\gamma\omega_n t}}{m\omega_d} \int_0^t e^{-(\alpha-\gamma\omega_n)\tau} . \{\sin \omega_d t . \cos \omega_d \tau - \cos \omega_d t \cdot \sin \omega_d \tau\}\, d\tau$$

$$= \frac{F_0\, e^{-\gamma\omega_n t}}{m\omega_d} \sin \omega_d t \left[\frac{e^{-(\alpha-\gamma\omega_n)\tau}}{(\alpha-\gamma\omega_n)^2 + \omega_d^2}\{-(\alpha-\gamma\omega_n)\cos \omega_d \tau + \omega_d \sin \omega_d \tau\}\right]_0^t$$

$$- \frac{F_0\, e^{-\gamma\omega_n t}}{m\omega_d} \cos \omega_d t \left[\frac{e^{-(\alpha-\gamma\omega_n)\tau}}{(\alpha-\gamma\omega_n)^2 + \omega_d^2}\{-(\alpha-\gamma\omega_n)\sin \omega_d \tau - \omega_d \cos \omega_d \tau\}\right]_0^t$$

$$= \frac{F_0\, e^{-\gamma\omega_n t}}{m\omega_d\left[(\alpha-\gamma\omega_n)^2 + \omega_d^2\right]}\left\{\omega_d\, e^{-(\alpha-\gamma\omega_n)t} + (\alpha-\gamma\omega_n)\sin \omega_d t - \omega_d \cos \omega_d t\right\}$$

$$= \frac{F_0\, e^{-\alpha t}}{m\left[(\alpha-\gamma\omega_n)^2 + \omega_d^2\right]} + \frac{F_0\, e^{-\gamma\omega_n t}}{m\omega_d\left[(\alpha-\gamma\omega_n)^2 + \omega_d^2\right]} \sin(\omega_d t - \phi)$$

where $\phi = \tan^{-1}\left(\frac{\omega_d}{\alpha - \gamma\omega_n}\right)$.

4.17 Equation of motion:

$$m\ddot{x} + k\,x = A\,p(t)$$

or $\quad 10\ddot{x} + 1000\,x = \frac{\pi}{4}(0.1)^2 (50)(1 - e^{-3t}) = 0.3927 - 0.3927\, e^{-3t}$

Solution:

$$x(t) = C_1 \cos \omega_n t + C_2 \sin \omega_n t + \frac{0.3927}{k} - \frac{0.3927}{k + m(3^2)} e^{-3t}$$

where $\omega_n = \sqrt{\frac{k}{m}} = \sqrt{\frac{1000}{10}} = 10$ rad/sec. Thus x(t) becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5}) - 36.0275\,(10^{-5})\,e^{-3t}\ \text{m}$$

where C_1 and C_2 can be determined from the initial conditions. As $t \to \infty$, $e^{-3t} \to 0$ and the steady state response becomes

$$x(t) = C_1 \cos 10\,t + C_2 \sin 10\,t + 39.27\,(10^{-5})\ \text{m}$$

4.18 For $0 \le t \le \frac{\pi}{\omega}$: Equation of motion: $m\ddot{x} + kx = \frac{F_0}{2} - \frac{F_0}{2}\cos\omega t$

$$x(t) = A\cos\omega_n t + B\sin\omega_n t + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\cos\omega t$$

$$x(0)=0 \quad \therefore \quad A + \frac{F_0}{2k} - \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}, \quad A = \frac{F_0\left(\frac{\omega}{\omega_n}\right)^2}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}$$

$$\dot{x}(0)=0 \quad \therefore \quad B = 0$$

$$x(t) = \frac{F_0}{2k\left\{1-\left(\frac{\omega}{\omega_n}\right)^2\right\}}\left[1-\cos\omega t - \left(\frac{\omega}{\omega_n}\right)^2\left(1-\cos\omega_n t\right)\right]$$

At $t = \frac{\pi}{\omega}$, $\quad x\left(\frac{\pi}{\omega}\right) = \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2-\frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right]$

For $t > \frac{\pi}{\omega}$: Eq. (4.33) gives $\quad x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$

$$x(t) = \frac{1}{m\omega_n}\int_0^{\pi/\omega} F(\tau)\cdot\sin\omega_n(t-\tau)\,d\tau + \frac{1}{m\omega_n}\int_{\pi/\omega}^{t} F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n}\int_{\pi/\omega}^{t}\sin\omega_n(t-\tau)\,d\tau$$

$$= x\left(t=\frac{\pi}{\omega}\right) + \frac{F_0}{m\omega_n^2}\Big[\cos\omega_n(t-\tau)\Big]_{\tau=\pi/\omega}^{t}$$

$$= \frac{F_0}{2k\left(1-\frac{\omega^2}{\omega_n^2}\right)}\left[2-\frac{\omega^2}{\omega_n^2}\left(1-\cos\frac{\omega_n\pi}{\omega}\right)\right] + \frac{F_0}{k}\left[1-\cos\omega_n\left(t-\frac{\pi}{\omega}\right)\right]$$

4.19 For an undamped system Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n}\int_0^t F(\tau)\sin\omega_n(t-\tau)\,d\tau$$

$$F(\tau) = \begin{cases} F_0 & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$$

For $0 \le t \le t_0$: $x(t) = \frac{F_0}{m\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{m\omega_n^2}(1 - \cos \omega_n t)$

$$= \frac{F_0}{k}\left(1 - \cos \omega_n t\right)$$

For $t > t_0$: $x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \sin \omega_n (t-\tau)\, d\tau = \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t_0}$

$$= \frac{F_0}{k}\left[\cos \omega_n (t-t_0) - \cos \omega_n t\right]$$

4.20 $F(\tau) = \begin{cases} F_0\left(\frac{\tau}{t_0}\right) & \text{for } 0 \le \tau \le t_0 \\ 0 & \text{for } \tau > t_0 \end{cases}$

For $0 \le t \le t_0$: $x(t) = \frac{F_0}{m\omega_n t_0} \int_0^t \tau \sin \omega_n (t-\tau)\, d\tau$

i.e. $x(t) = \frac{F_0}{m\omega_n t_0}\left[\int_0^t (t-\tau) \sin \omega_n (t-\tau)(-d\tau) - t \int_0^t \sin \omega_n (t-\tau)(-d\tau)\right]$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos \omega_n (t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k\, t_0}\left(t - \frac{1}{\omega_n} \sin \omega_n t\right)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n t_0} \int_0^{t_0} \tau \sin \omega_n (t-\tau)\, d\tau$$

$$= \frac{F_0}{m\omega_n t_0}\left[\frac{1}{\omega_n^2} \sin \omega_n (t-\tau) - \frac{(t-\tau)}{\omega_n} \cos \omega_n (t-\tau)\right]_{\tau=0}^{t}$$

$$+ \frac{F_0\, t}{m\omega_n t_0}\left[\frac{\cos \omega_n (t-\tau)}{\omega_n}\right]_{\tau=0}^{t}$$

$$= \frac{F_0}{k\, t_0}\left[\frac{1}{\omega_n} \sin \omega_n (t-t_0) + t_0 \cos \omega_n (t-t_0) - \frac{1}{\omega_n} \sin \omega_n t\right]$$

4.21 $F(t) = \begin{cases} F_0\left(1 - \cos \frac{\pi t}{2 t_0}\right) & ; \ 0 \le t \le t_0 \\ 0 & ; \ t > t_0 \end{cases}$ $\quad (E_1)$

For an undamped system, Eq. (4.33) gives

$$x(t) = \frac{1}{m\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau$$

For $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^t \left(1 - \cos\frac{\pi\tau}{2t_0}\right) \sin\omega_n(t-\tau)\, d\tau \qquad (E_2)$$

Noting that

$$\int_0^t \sin(\omega_n t - \omega_n \tau)\, d\tau = \left[\frac{1}{\omega_n} \cos(\omega_n t - \omega_n \tau)\right]_0^t = \frac{1}{\omega_n}(1 - \cos\omega_n t)$$

and

$$\int_0^t \cos\frac{\pi\tau}{2t_0} \left(\sin\omega_n t \cos\omega_n \tau - \cos\omega_n t \sin\omega_n \tau\right) d\tau$$

$$= \sin\omega_n t \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right]_0^t$$

$$-\cos\omega_n t \left[-\frac{\cos\left(\omega_n - \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{\cos\left(\omega_n + \frac{\pi}{2t_0}\right)\tau}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]_0^t$$

$$= \frac{-1}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} \left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] + \frac{1}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right],$$

Eq. (E_2) can be simplified as

$$x(t) = \frac{F_0}{k}(1 - \cos\omega_n t) + \frac{(F_0/m)}{\left\{\left(\frac{\pi}{2t_0}\right)^2 - \omega_n^2\right\}} \left[\cos\frac{\pi t}{2t_0} - \cos\omega_n t\right] \qquad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{m\omega_n} \int_0^{t_0} \left(1 - \cos\frac{\pi\tau}{2t_0}\right) \sin\omega_n(t-\tau)\, d\tau \qquad (E_4)$$

$$= \frac{F_0}{m\omega_n}\left[\frac{1}{\omega_n}\cos\omega_n(t-t_0) - \frac{1}{\omega_n}\cos\omega_n t\right] - \frac{F_0}{m\omega_n}\left\{\sin\omega_n t \, * \right.$$

$$\left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\frac{\pi}{2t_0} + \omega_n\right)}\right] + \cos\omega_n t \, *$$

$$\left.\left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t_0}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t_0}{2\left(\omega_n + \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n - \frac{\pi}{2t_0}\right)} - \frac{1}{2\left(\omega_n + \frac{\pi}{2t_0}\right)}\right]\right\}$$

i.e., $x(t) = \frac{F_0}{k}\left[\cos\omega_n(t - t_0) - \cos\omega_n t\right]$

$$- \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o} - \omega_n\right)} \sin \omega_n (t - t_o) - \frac{F_o}{2m\omega_n\left(\frac{\pi}{2t_o} + \omega_n\right)} \sin \omega_n (t - t_o)$$

$$+ \frac{F_o}{2m\omega_n} \cos \omega_n t \cdot \left\{ - \frac{1}{\left(\frac{\pi}{2t_o} - \omega_n\right)} + \frac{1}{\left(\frac{\pi}{2t_o} + \omega_n\right)} \right\} \qquad (E_5)$$

4.22 Base displacement $= y(s) = Y \sin \frac{\pi s}{\delta}$ $\qquad (E_1)$

i.e., $\qquad y(t) = Y \sin \frac{\pi t}{t_o}$ $\qquad (E_2)$

steady-state relative displacement can be found from Eq. (4.36) as

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta\omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_3)$$

where $\qquad \ddot{y}(\tau) = -Y\left(\frac{\pi}{t_o}\right)^2 \sin \frac{\pi\tau}{t_o}$ $\qquad (E_4)$

$$z(t) = \frac{Y}{\omega_d}\left(\frac{\pi}{t_o}\right)^2 \int_0^t e^{-\zeta\omega_n t}\, e^{\zeta\omega_n\tau} \sin \frac{\pi\tau}{t_o} \sin \omega_d (t-\tau)\, d\tau \qquad (E_5)$$

But $\quad \sin \frac{\pi\tau}{t_o} \sin \omega_d (t-\tau) = \frac{1}{2}\cos\left(\frac{\pi\tau}{t_o} - \omega_d t + \omega_d\tau\right) - \frac{1}{2}\cos\left(\frac{\pi\tau}{t_o} + \omega_d t - \omega_d\tau\right)$

$$= \frac{1}{2}\left[\cos\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot \cos\omega_d t + \sin\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot \sin\omega_d t\right]$$

$$-\frac{1}{2}\left[\cos\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot \cos\omega_d t - \sin\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot \sin\omega_d t\right] \qquad (E_6)$$

Eqs. (E_5) and (E_6) give

$$z(t) = \frac{1}{2}\frac{Y}{\omega_d}\left(\frac{\pi}{t_o}\right)^2 e^{-\zeta\omega_n t}\left[\cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cos\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot d\tau\right.$$

$$+ \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_o} + \omega_d\right)\tau \cdot d\tau$$

$$- \cos\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \cos\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot d\tau$$

$$\left. + \sin\omega_d t \cdot \int_0^t e^{\zeta\omega_n\tau} \cdot \sin\left(\frac{\pi}{t_o} - \omega_d\right)\tau \cdot d\tau\right] \qquad (E_7)$$

Eq. (E_7) can be simplified

$$z(t) = \frac{Y}{2\omega_d}\left(\frac{\pi}{t_o}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_o} + \omega_d\right)^2}\left\{\zeta\omega_n \cos\frac{\pi t}{t_o}\right.$$

$$+ \left(\frac{\pi}{t_o} + \omega_d\right)\sin\frac{\pi t}{t_o} - \zeta\omega_n\, e^{-\zeta\omega_n t} \cos\omega_d t$$

$$\left. + \left(\frac{\pi}{t_o} + \omega_d\right) e^{-\zeta\omega_n t} \sin\omega_d t\right\}$$

$$+ \frac{Y}{2\omega_d}\left(\frac{\pi}{t_o}\right)^2 \frac{1}{(\zeta\omega_n)^2 + \left(\frac{\pi}{t_o} - \omega_d\right)^2}\left\{-\zeta\omega_n \cos\frac{\pi t}{t_o}\right.$$

$$- \left(\frac{\pi}{t_0} - \omega_d \right) \sin \frac{\pi t}{t_0} + \zeta \omega_n \, e^{-\zeta \omega_n t} \cos \omega_d t$$

$$+ \left(\frac{\pi}{t_0} - \omega_d \right) e^{-\zeta \omega_n t} \sin \omega_d t \Big\} \qquad (E_8)$$

4.23 Base displacement:

$$y(t) = \begin{cases} \dfrac{Y\,v\,t}{\delta} \;;\; 0 \le t \le t_0 = \dfrac{\delta}{v} \\ 0 \;;\; t > t_0 = \dfrac{\delta}{v} \end{cases} \qquad (1)$$

Equation of motion of vehicle:

$$m\,\ddot{x} + k\,(x - y) = 0 \qquad (2)$$

Using Eq. (1), Eq. (2) can be expressed as

$$m\,\ddot{x} + k\,x = k\,y = \begin{cases} \dfrac{k\,v\,Y\,t}{\delta} \;;\; 0 \le t \le t_0 \\ 0 \;;\; t > t_0 \end{cases} \qquad (3)$$

Steady state solution of Eq. (3) from Example 4.6:

$$x(t) = \begin{cases} \dfrac{v\,Y}{\delta\,\omega_n\,k} \left\{ \omega_n\,t - \sin \omega_n\,t \right\} \;;\; 0 \le t \le t_0 \\ 0 \;;\; t > t_0 \end{cases} \qquad (4)$$

Note that the homogeneous solution,

$$x(t) = C_1 \cos \omega_n\,t + C_2 \sin \omega_n\,t \qquad (5)$$

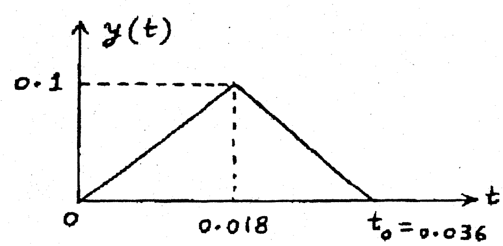
is to be added to Eq. (4) to obtain the complete solution. The constants C_1 and C_2 are to be evaluated from the initial conditions (at $t = 0$). In fact, tthe resulting complete solution is valid for all values of t, including values of $t > t_0$.

4.24 Speed of automobile = 50 km/hr

Excitation frequency $= \left(\dfrac{50 \times 1000}{3600} \right) \dfrac{1}{0.5} = 27.7778$ Hz

Natural frequency $= f_n = 1.0$ Hz $\Rightarrow \omega_n = 2\pi$ rad/sec

$$t_0 = \frac{0.5 \times 3600}{50 \times 1000} = 0.036 \text{ sec}$$



$$y(t) = \begin{cases} \dfrac{0.2\,t}{t_0} \;;\; 0 \le t \le t_0/2 \\ -\dfrac{0.2\,t}{t_0} + 0.2 \;;\; \dfrac{t_0}{2} \le t \le t_0 \\ 0 \;;\; t > t_0 \end{cases} \quad \cdots (E_1)$$

Equation of motion (for undamped case):

$$m\ddot{x} + k(x-y) = 0 \qquad \text{or} \qquad m\ddot{x} + kx = ky = F(t) \tag{E_2}$$

$$\text{where} \qquad F(t) = k\, y(t) \tag{E_3}$$

Solution of Eq. (E_2) is:

$$x(t) = \frac{1}{m\,\omega_n} \int_0^t F(\tau)\, \sin \omega_n (t-\tau)\, d\tau \tag{E_4}$$

For $0 \le t \le \frac{t_0}{2}$:

$$x(t) = \frac{k}{m\,\omega_n} \int_0^t \left(\frac{0.2}{t_0}\right) \tau \, \sin \omega_n (t-\tau)\, d\tau \tag{E_5}$$

Since $\int_0^t \tau \sin \omega_n (t-\tau)\, d\tau = \left(\frac{t}{\omega_n} - \frac{1}{\omega_n^2} \sin \omega_n t\right)$,

Eq. (E_5) becomes

$$x(t) = 5.5556\,(t - 0.1592 \sin 6.2832\, t) \ \text{m} \quad ; \ 0 \le t \le 0.018 \text{ sec} \tag{E_6}$$

For $\frac{t_0}{2} \le t \le t_0$:

$$x(t) = \frac{k}{m\,\omega_n} \left\{ \int_0^{t_0/2} \frac{0.2\,\tau}{t_0} \sin \omega_n (t-\tau)\, d\tau + \int_{t_0/2}^{t} \left(-\frac{0.2\,\tau}{t_0} + 0.2\right) \sin \omega_n (t-\tau)\, d\tau \right\}$$

But

$$\frac{0.2\,k}{m\,\omega_n\, t_0} \cdot \int_0^{t_0/2} \tau \cdot \sin \omega_n (t-\tau)\, d\tau = \frac{0.2\,k}{m\,\omega_n\, t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_0^{t_0/2} - \cos \omega_n t \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_0^{t_0/2} \right\}$$

$$= 5.5556 \Big[0.1592 \sin 6.2832\,(t - 0.018) + 0.0180 \cos 6.2832\,(t - 0.018) - 0.1592 \sin 6.2832\, t \Big] \ \text{m} \tag{E_7}$$

since $t_0 = 0.036$.

$$-\frac{0.2\,k}{m\,\omega_n\, t_0} \int_{t_0/2}^{t} \tau \sin \omega_n (t-\tau)\, d\tau = -\frac{0.2\,k}{m\,\omega_n\, t_0} \left\{ \sin \omega_n t \left[\frac{1}{\omega_n^2} \cos \omega_n \tau + \frac{\tau}{\omega_n} \sin \omega_n \tau \right]_{t_0/2}^{t} - \cos \omega_n t \cdot \left[\frac{1}{\omega_n^2} \sin \omega_n \tau - \frac{\tau}{\omega_n} \cos \omega_n \tau \right]_{t_0/2}^{t} \right\}$$

$$= -5.5556 \Big[t - 0.1592 \sin 6.2832\,(t - 0.018) - 0.0180 \cos 6.2832\,(t - 0.018) \Big] \ \text{m} \tag{E_8}$$

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t} \sin \omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos \omega_n(t-\tau)\right]_{t_0/2}^{t}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[1-\cos \omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[1-\cos 6.2832\,(t-0.018)\right] \text{ m} \qquad (E_9)$$

Hence the solution can be expressed as

$$x(t) = \left[1.7689 \sin 6.2832\,(t-0.018) - 0.8845 \sin 6.2832\,t \right.$$
$$\left. - 5.5556\,t + 0.2\right] \text{ m} \; ; \qquad 0.018 \le t \le 0.036 \text{ sec} \qquad (E_{10})$$

For $t > t_0$:

$$x(t) = \frac{0.2\,k}{m\,\omega_n t_0}\int_0^{t_0/2} \tau \sin \omega_n(t-\tau)\,d\tau - \frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau)\,d\tau$$
$$+ \frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0} \sin \omega_n(t-\tau)\,d\tau \qquad (E_{11})$$

The first term on the right side of (E_{11}) is given by (E_7).

Second term on the right side of (E_{11}) is

$$-\frac{0.2\,k}{m\,\omega_n t_0}\int_{t_0/2}^{t_0} \tau \sin \omega_n(t-\tau)\,d\tau = -\frac{0.2\,k}{m\,\omega_n t_0}\left\{\sin \omega_n t \cdot\left[\frac{1}{\omega_n^2}\cos \omega_n \tau \right.\right.$$
$$\left.\left. + \frac{\tau}{\omega_n}\sin \omega_n \tau\right]_{t_0/2}^{t_0} - \cos \omega_n t \cdot \left[\frac{1}{\omega_n^2}\sin \omega_n \tau - \frac{\tau}{\omega_n}\cos \omega_n \tau\right]_{t_0/2}^{t_0}\right\}$$

$$= -5.5556\left[0.1592 \sin 6.2832\,(t-0.036) + 0.0360 \cos 6.2832\,(t-0.036)\right.$$
$$\left. - 0.1592 \sin 6.2832\,(t-0.018) - 0.0180 \cos 6.2832\,(t-0.018)\right] \text{ m} \qquad (E_{12})$$

The third term on the right side of (E_{11}) is:

$$\frac{0.2\,k}{m\,\omega_n}\int_{t_0/2}^{t_0} \sin \omega_n(t-\tau)\,d\tau = \frac{0.2\,k}{m\,\omega_n}\left[\frac{1}{\omega_n}\cos \omega_n(t-\tau)\right]_{t_0/2}^{t_0}$$

$$= \frac{0.2\,k}{m\,\omega_n^2}\left[\cos \omega_n(t-t_0) - \cos \omega_n\left(t-\frac{t_0}{2}\right)\right]$$

$$= 0.2\left[\cos 6.2832\,(t-0.036) - \cos 6.2832\,(t-0.018)\right] \text{ m} \qquad (E_{13})$$

$\therefore$ $x(t)$ is given by the sum of Eqs. (E_7), (E_{12}) and (E_{13}), which can be simplified as

$$x(t) = 1.7689 \sin 6.2832\,(t - 0.018) - 0.8845 \sin 6.2832\, t - 0.8845 \sin 6.2832\,(t - 0.036) \text{ m} \; ; \quad t > 0.036 \text{ sec} \qquad (E_{14})$$

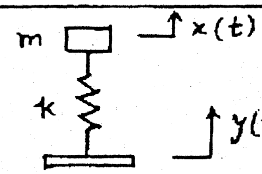
4.25 When the container strikes the floor, the velocity of the mass is given by

$$m g h = \tfrac{1}{2} m v^2 \quad \text{or} \quad v = \sqrt{2 g h} \qquad (E_1)$$

The displacement of the camcorder subjected to an initial velocity $\dot{x}_0 = v$ is given by Eq. (2.72), with $x_0 = 0$ and $\zeta < 1$,

$$x(t) = e^{-\zeta \omega_n t} \cdot \frac{\dot{x}_0}{\omega_n \sqrt{1-\zeta^2}} \cdot \sin \sqrt{1-\zeta^2}\, \omega_n t \qquad (E_2)$$

4.26 System can be modeled as a spring-mass system subjected to base motion:



$$y(t) = \begin{cases} (Y t^2 / t_0^2) \; ; & 0 \le t \le t_0 \\ 0 \; ; & t > t_0 \end{cases} \qquad --- (E_1)$$

Relative displacement of mass, $z = x - y$, is given by Eq. (4.36):

$$z(t) = -\frac{1}{\omega_d} \int_0^t \ddot{y}(\tau)\, e^{-\zeta \omega_n (t-\tau)} \sin \omega_d (t-\tau)\, d\tau \qquad (E_2)$$

where

$$\left.\begin{aligned} y(\tau) &= (Y \tau^2 / t_0^2) \\ \ddot{y}(\tau) &= (2 Y / t_0^2) \end{aligned}\right\} \; ; \quad 0 \le \tau \le t_0 \qquad (E_3)$$

and

$$y(\tau) = 0 \; ; \quad \tau > t_0$$

For $0 \le t \le t_0$:

Since the system is undamped, $\omega_d = \omega_n$ and $\zeta = 0$, and (E_2) reduces to

$$z(t) = -\frac{2 Y}{\omega_n t_0^2} \int_0^t \sin \omega_n (t-\tau)\, d\tau \qquad (E_4)$$

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \int_0^t (\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau)\, d\tau$$

$$= \sin \omega_n t \int_0^t \cos \omega_n \tau\, d\tau - \cos \omega_n t \int_0^t \sin \omega_n \tau\, d\tau$$

$$= \sin \omega_n t \left(\frac{1}{\omega_n} \sin \omega_n \tau \right)_0^t - \cos \omega_n t \left(-\frac{1}{\omega_n} \cos \omega_n \tau \right)_0^t$$

$$= \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \qquad (E_5)$$

Thus Eq. (E_4) gives

$$x(t) = \frac{Y\, t^2}{t_0^2} - \frac{2\,Y}{t_0^2\, \omega_n^2} \left(1 - \cos \omega_n t \right) \; ; \quad 0 \le t \le t_0 \qquad (E_6)$$

For $t > t_0$:

Eq. (E_2) gives

$$z(t) = -\frac{1}{\omega_n} \int_0^{t_0} \frac{2Y}{t_0^2} \sin \omega_n (t - \tau)\, d\tau \qquad (E_7)$$

But

$$\int_0^{t_0} \sin \omega_n (t - \tau)\, d\tau = \frac{1}{\omega_n} \left\{ \cos \omega_n (t - t_0) - \cos \omega_n t \right\}$$

$$\therefore \; z(t) = x(t) - y(t) = -\frac{2\,Y}{\omega_n^2\, t_0^2} \left\{ \cos \omega_n (t - t_0) - \cos \omega_n t \right\} ; \quad t > t_0 \qquad (E_8)$$

4.27 $$I_1 = \int_0^t (t - \tau)\, e^{-\zeta \omega_n (t - \tau)} \sin \omega_d (t - \tau) (-d\tau)$$

$$= \left[\frac{(t - \tau)\, e^{-\zeta \omega_n (t - \tau)}}{(\zeta \omega_n)^2 + \omega_d^2} \left\{ -\zeta \omega_n \sin \omega_d (t - \tau) - \omega_d \cos \omega_d (t - \tau) \right\} \right.$$

$$\left. - \frac{e^{-\zeta \omega_n (t - \tau)}}{(\zeta \omega_n)^2 + \omega_d^2} \left\{ \left(\zeta^2 \omega_n^2 - \omega_d^2 \right) \sin \omega_d (t - \tau) + 2 \zeta \omega_n \omega_d \cos \omega_d (t - \tau) \right\} \right]_{\tau = 0}^{t}$$

$$= -\frac{1}{\omega_n^4} (2 \zeta \omega_n \omega_d) + \frac{t\, e^{-\zeta \omega_n t}}{\omega_n^2} \left(\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t \right)$$

$$+ \frac{e^{-\zeta \omega_n t}}{\omega_n^4} \left\{ \omega_n^2 (2 \zeta^2 - 1) \sin \omega_d t + 2 \zeta \omega_n \omega_d \cos \omega_d t \right\}$$

$$I_2 = \int_0^t e^{-\zeta \omega_n (t - \tau)} \sin \omega_d (t - \tau) (-d\tau)$$

$$= \left[\frac{e^{-\zeta \omega_n (t - \tau)}}{\zeta^2 \omega_n^2 + \omega_d^2} \left\{ -\zeta \omega_n \sin \omega_d (t - \tau) - \omega_d \cos \omega_d (t - \tau) \right\} \right]_{\tau = 0}^{t}$$

$$= -\frac{\omega_d}{\omega_n^2} + \frac{e^{-\zeta \omega_n t}}{\omega_n^2} \left(\zeta \omega_n \sin \omega_d t + \omega_d \cos \omega_d t \right)$$

$$x(t) = \frac{\delta F}{m\,\omega_d}\cdot I_1 - \frac{\delta F \cdot t}{m\,\omega_d}\cdot I_2$$

$$= \frac{\delta F}{k}\left[t - \frac{2\zeta}{\omega_n} + e^{-\zeta \omega_n t}\left\{ \frac{2\zeta}{\omega_n}\cos\omega_d t - \left(\frac{\omega_d^2-\omega_n^2}{\omega_n^2\,\omega_d}\right)\sin\omega_d t \right\}\right]$$

4.28 (1) $m\ddot{x} + c\dot{x} + k x = F(t)$

where $m(t) = M - m_0 t = (2000 - 10t)$ kg

and $F(t) = m_0 v = 10(2000) = 20000$ N

(2) With $m = M - \frac{1}{2} m_0 t_0 = 2000 - \frac{1}{2}(10)(100) = 1500$ kg, equation of motion becomes

$$1500\,\ddot{x} + 0.1\times10^6\,\dot{x} + 7.5\times10^6\,x = 20\,000 = \text{constant}$$

Maximum steady state displacement is

$$x_p(t) = \frac{F}{k} = \frac{20000}{7.5\times10^6} = 0.002667 \text{ m}$$

4.29 From Eq.(4.32), the response to unit step function can be obtained by setting $F(\tau) = 1$ as

$$h(t) = \int_0^t g(t-\tau)\,d\tau \qquad \text{----}(E_1)$$

By differentiating this equation with respect to t, we obtain

$$\frac{dh}{dt}(t) = g(t)$$

4.30 Equation (4.32) gives $x(t) = \int_0^t F(\tau)\cdot g(t-\tau)\cdot d\tau$

But $g(t-\tau) = \frac{dh}{d\tau}(t-\tau)$ from problem 4.29.

$$x(t) = \int_0^t F(\tau)\,\frac{dh}{d\tau}(t-\tau)\,d\tau$$

Integration by parts gives

$$x(t) = -F(t)\cdot h(t-\tau)\Big|_{\tau=0}^{t} + \int_0^t \frac{dF}{d\tau}\cdot h(t-\tau)\,d\tau$$

$$= -F(t)\,h(0) + F(0)\,h(t) + \int_0^t \frac{dF}{d\tau}\,h(t-\tau)\,d\tau$$

But $h(0) = 0$ from Eq. (E_1) of problem 4.29.

$$\therefore\; x(t) = F(0)\,h(t) + \int_0^t \frac{dF}{d\tau}(\tau)\cdot h(t-\tau)\cdot d\tau$$

4.31 Equation of motion for rotation about O:

$$J_0\,\ddot{\theta} + M\,\ddot{x}\,(\ell) + k_1\,a^2\,\theta + k_2\,b^2\,\theta = F_0\,\ell\,e^{-t} \qquad (1)$$

where $\ddot{x} = \ell\,\ddot{\theta}$ and

$$J_0 = \frac{1}{12}\,m\,\ell^2 + m\left(\frac{\ell}{2}\right)^2 = \frac{1}{3}\,m\,\ell^2$$

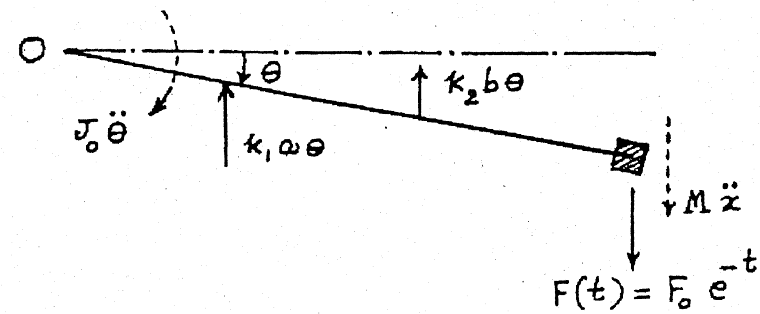
Eq. (1) can be rewritten as:

$$\left(\frac{1}{3}\, m\, \ell^2 + M\, \ell^2\right) \ddot{\theta} + (k_1\, a^2 + k_2\, b^2)\, \theta = F_0\, \ell\, e^{-t} \tag{2}$$

For given data, Eq. (2) takes the form:

$$53.3333\, \ddot{\theta} + 1562.5\, \theta = 500\, e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\dfrac{1562.5}{53.3333}} = 5.4127$ rad/sec and



the forcing term as $500\, e^{-t}$, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(53.3333)\,(5.4127)} \int_0^t 500\, e^{-\tau} \sin 5.4127\,(t-\tau)\, d\tau$$

$$= 1.7320 \int_0^t e^{-t}\, e^{(t-\tau)} \sin 5.4127\,(t-\tau)\, d\tau$$

$$= -\,1.7320\, e^{-t} \int_0^t e^{(t-\tau)} \sin 5.4127\,(t-\tau)\,(-\,d\tau) \tag{4}$$

Using the formula:

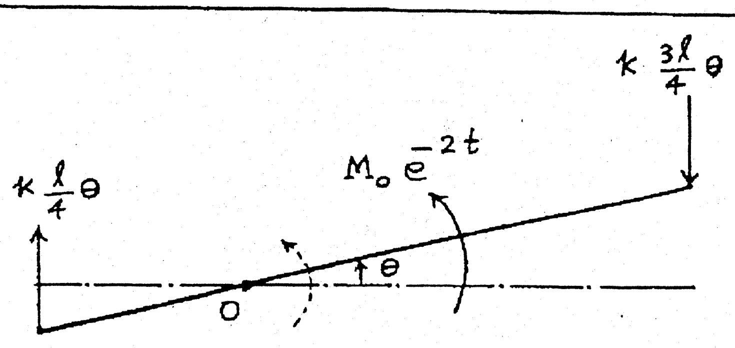
$$\int e^{ax} \sin b\,x\, dx = \frac{1}{a^2 + b^2}\, e^{ax} \left[a \sin b\,x - b \cos b\,x\right] \tag{5}$$

Eq. (4) can be expressed as

$$\theta(t) = -\,1.7320\, e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 5.4127^2} \left\{\sin 5.4127\,(t-\tau) - 5.4127 \cos 5.4127\,(t-\tau)\right\}\right]_{\tau=0}^{\tau=t}$$

$$= 0.3094\, e^{-t} + 0.05717 \sin 5.4127\, t - 0.3094 \cos 5.4127\, t \text{ radian}$$

4.32



$$J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583 \text{ kg–m}^2$$

Equation of motion for rotation about O:

$$J_0 \ddot{\theta} + k \frac{\ell^2}{16} \theta + k \frac{9}{16} \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = M_0 e^{-2t}$$

$$\text{or} \quad 1.4583 \ddot{\theta} + 3125.0 \theta = 100 e^{-2t} \qquad (1)$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915$ rad/sec, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$\theta(t) = \frac{1}{(1.4583)(46.2915)} \int_0^t (100 e^{-2\tau}) \sin 46.2915 (t - \tau) \, d\tau$$

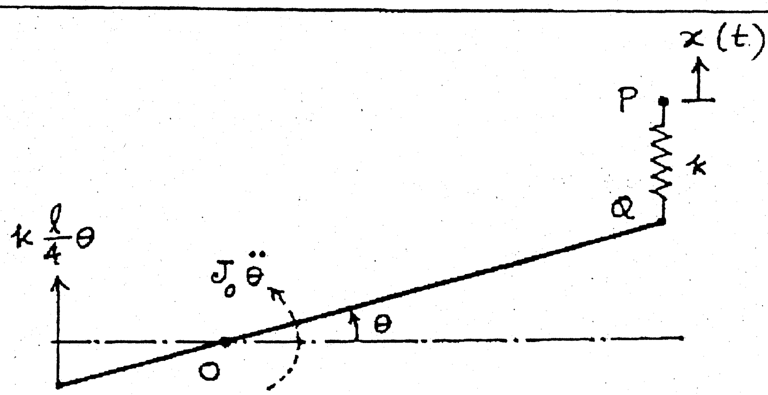
$$= -1.4813 e^{-2t} \int_0^t e^{2(t-\tau)} \sin 46.2915 (t-\tau) (-d\tau) \qquad (2)$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (2) can be expressed as:

$$\theta(t) = -1.4813 e^{-2t} \left[\frac{e^{2(t-\tau)}}{2^2 + 46.2915^2} \left\{ 2 \sin 46.2915 (t-\tau) - 46.2915 \cos 46.2915 (t-\tau) \right\} \right]_{\tau=0}^{\tau=t}$$

$$= 0.03194 e^{-2t} + 13.7994 (10^{-4}) \sin 46.2915 t - 0.03194 \cos 46.2915 t \text{ radian} \qquad (4)$$

4.33



Net compression of spring PQ $= \frac{3 \ell \theta}{4} - x(t)$. Equation of motion for rotation about O:

$$J_0 \ddot{\theta} = -k \frac{\ell \theta}{4} \cdot \left(\frac{\ell}{4}\right) - k \left(\frac{3 \ell \theta}{4} - x(t)\right) \left(\frac{3 \ell}{4}\right)$$

$$\text{or} \quad J_0 \ddot{\theta} + \frac{5}{8} k \ell^2 \theta = \frac{3 k \ell}{4} x(t) = \frac{3 k \ell}{4} x_0 e^{-t} \qquad (1)$$

For given data, $J_0 = \frac{1}{12} m \ell^2 + m \left(\frac{\ell}{4}\right)^2 = \frac{7}{48} m \ell^2 = \frac{7}{48} (10)(1^2) = 1.4583$ kg–m^2, $\frac{5}{8} k \ell^2 = \frac{5}{8} (5000)(1^2) = 3125$ N/m, and Eq. (1) becomes:

$$1.4583 \ddot{\theta} + 3125.0 \theta = 37.5 e^{-t} \qquad (2)$$

Noting that the system is undamped with

$$\omega_n = \sqrt{\frac{3125.0}{1.4583}} = 46.2915 \text{ rad/sec}$$

the convolution integral, Eq. (4.33), can be used to find the steady state response as:

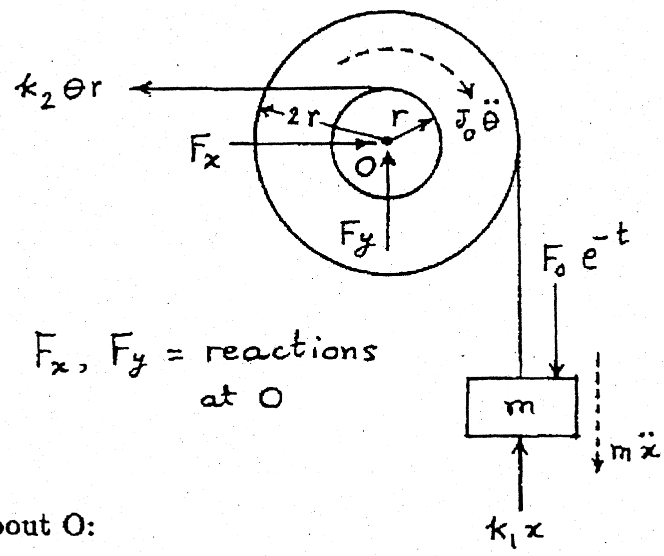
$$\theta(t) = \frac{1}{(1.4583)\,(46.2915)} \int_0^t (37.5\ e^{-\tau}) \sin 46.2915\,(t-\tau)\, d\tau \tag{3}$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (3) can be expressed as

$$\theta(t) = -\,0.5555\ e^{-t} \left[\frac{e^{(t-\tau)}}{1^2 + 46.2915^2} \left\{ \sin 46.2915\,(t-\tau) - 46.2915 \cos\,(t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.01199\ e^{-t} + 2.591\,(10^{-4}) \sin 46.2915\ t - 0.01199 \cos 46.2915\ t \text{ radian} \tag{4}$$

4.34



Equation of motion for rotation of pulley about O:

$$J_0\, \ddot{\theta} + m\, \ddot{x}\,(2\,r) + k_1\, x\,(2\,r) + k_2\,(\theta\, r)\, r = 2\,r\,F_0\, e^{-t} \tag{1}$$

where $\theta = \frac{x}{2\,r}$. Eq. (1) can be rewritten in terms of x only as:

$$\left(\frac{J_0}{2\,r} + 2\,m\,r \right) \ddot{x} + \left(2\,k_1\,r + \frac{1}{2}\,k_2\,r \right) x = 2\,r\,F_0\, e^{-t} \tag{2}$$

For given data, Eq. (2) becomes

$$11.0\,\ddot{x} + 112.5\,x = 5\,e^{-t} \tag{3}$$

Noting that the system is undamped with $\omega_n = \sqrt{\frac{112.5}{11.0}} = 3.1980$ rad/sec, the convolution integral, Eq. (4.33), can be used to find the steady state response as:

$$x(t) = -\,0.1421\ e^{-t} \int_0^t e^{(t-\tau)} \sin 3.1980\,(t-\tau))\,(-\,d\tau) \tag{4}$$

Using Eq. (5) in the solution of Problem 4.31, Eq. (4) can be expressed as

$$x(t) = -\frac{0.1421\ e^{-t}}{1^2 + 3.1980^2}\left[e^{(t-\tau)} \left\{ \sin 3.1980\ (t-\tau) - 3.1980 \cos 3.1980\ (t-\tau) \right\} \right]_{\tau=0}^{t}$$

$$= 0.04048\ e^{-t} + 0.01266 \sin 3.1980\ t - 0.04048 \cos 3.1980\ t \text{ m} \qquad (5)$$

4.35
$$F(t) = \begin{cases} F_o & ; \quad 0 \le t \le t_o \\ 0 & ; \quad t > t_o \end{cases} \qquad (E_1)$$

Eq. (4.33) gives, for an undamped system,

$$x(t) = \frac{1}{m\,\omega_n} \int_0^t F(\tau) \sin \omega_n (t-\tau)\, d\tau \qquad (E_2)$$

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{m\,\omega_n} \int_0^t \sin \omega_n (t-\tau)\, d\tau = \frac{F_o}{k}\left(1 - \cos \omega_n t\right) \qquad (E_3)$$

using Eq. (E_5) in the solution of problem 4.26.

For $t > t_o$:

$$x(t) = \frac{F_o}{m\,\omega_n} \int_0^{t_o} \sin \omega_n (t-\tau)\, d\tau$$

Using the relation

$$\int_0^{t_o} \sin \omega_n (t-\tau) \cdot d\tau = \frac{1}{\omega_n}\left\{ \cos \omega_n (t-t_o) - \cos \omega_n t \right\}$$

the solution can be expressed as

$$x(t) = \frac{F_o}{k}\left[\cos \omega_n (t-t_o) - \cos \omega_n t \right] \qquad (E_4)$$

Response spectrum:

For $0 \le t \le t_o$, $\quad x(t) = \frac{F_o}{k}\left(1 - \cos \omega_n t\right) \qquad (E_5)$

$$\frac{dx}{dt} = \frac{F_o\,\omega_n}{k} \sin \omega_n t = 0 \Rightarrow \omega_n t_{max} = \pi$$

$$\therefore x_{max} = x(t = t_{max}) = \frac{F_o}{k}\left(1 - \cos \omega_n t_{max}\right) = \frac{2\,F_o}{k} \qquad (E_6)$$

For $t > t_o$,

$$x(t) = \frac{F_o}{k}\left\{ \cos \omega_n (t-t_o) - \cos \omega_n t \right\} \qquad (E_7)$$

$$\frac{dx}{dt} = -\frac{F_o\,\omega_n}{k}\left\{ \sin \omega_n (t-t_o) - \sin \omega_n t \right\} = 0$$

$$\Rightarrow \quad \sin \omega_n (t_{max} - t_0) = \sin \omega_n t_{max}$$

i.e.,
$$\tan \omega_n t_{max} = \left(\frac{\sin \omega_n t_0}{\cos \omega_n t_0 - 1} \right) \tag{E8}$$

i.e.,
$$\sin \omega_n t_{max} = \frac{\sin \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \tag{E9}$$

and
$$\cos \omega_n t_{max} = \frac{\cos \omega_n t_0 - 1}{\sqrt{2(1 - \cos \omega_n t_0)}} \tag{E10}$$

$$\therefore \quad x_{max} = x(t = t_{max}) = \frac{F_0}{k} \{ \cos \omega_n t_{max} \cdot \cos \omega_n t_0 + \sin \omega_n t_{max} \cdot \sin \omega_n t_0 - \cos \omega_n t_{max} \}$$

$$= \frac{F_0}{k} \left\{ \frac{(\cos \omega_n t_0 - 1)^2}{\sqrt{2(1 - \cos \omega_n t_0)}} + \frac{\sin^2 \omega_n t_0}{\sqrt{2(1 - \cos \omega_n t_0)}} \right\}$$

$$= \frac{2 F_0}{k} \sin \frac{\omega_n t_0}{2} \tag{E11}$$

Plotting: $\omega_n = \frac{2\pi}{\tau_n}$

For $0 \le t \le t_0$, $\omega_n t_{max} = \pi$ or $\frac{t_{max}}{\tau_n} = \frac{1}{2}$

when $t \le t_0$, $t_{max} = \frac{\tau_n}{2} \le t_0$

i.e., $\frac{t_0}{\tau_n} \ge \frac{1}{2}$

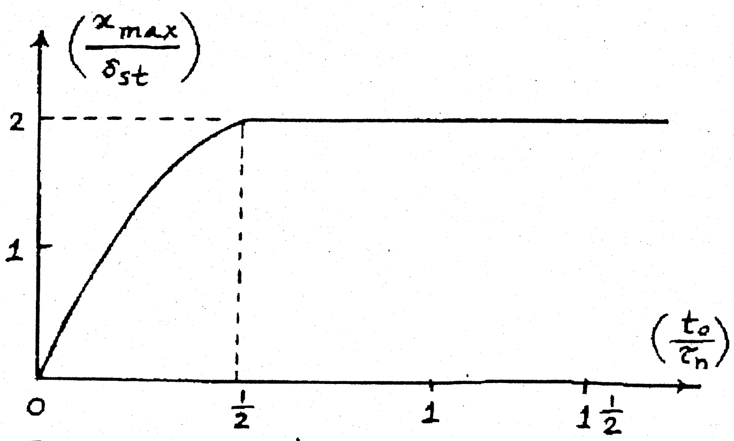
$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \quad \text{for} \quad \frac{t_0}{\tau_n} \ge \frac{1}{2} \tag{E12}$$

For $t > t_0$, $\frac{t_0}{\tau_n} < \frac{1}{2}$

$$\frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\omega_n t_0}{2} = 2 \sin \frac{2\pi t_0}{2 \tau_n}$$

$$\therefore \quad \frac{x_{max}}{\delta_{st}} = 2 \sin \frac{\pi t_0}{\tau_n} \quad \text{for} \quad \frac{t_0}{\tau_n} < \frac{1}{2} \tag{E13}$$

Eqs. (E_{12}) and (E_{13}) are plotted in the figure.



Response spectrum for a rectangular pulse-type load

(4.36) The response is found in problem 4.21.

For $0 \le t \le t_o$:

$$x(t) = \frac{F_o}{k}\left(1 - \cos\omega_n t\right) + \frac{F_o}{2 m \omega_n} \cdot \frac{2\omega_n}{\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}} \cdot \left\{\cos\frac{\pi t}{2t_o} - \cos\omega_n t\right\} \qquad (E_1)$$

For x_{max}, $\frac{dx}{dt} = 0$

i.e.,

$$\frac{F_o\,\omega_n}{k}\sin\omega_n t + \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(-\frac{\pi}{2t_o}\sin\frac{\pi t}{2t_o} + \omega_n \sin\omega_n t\right) = 0$$

which can be reduced to the form

$$\sin\omega_n t_{max} = \left[\frac{\pi}{2\omega_n t_o\left\{m\left(\frac{\pi}{2t_o}\right)^2 - m\,\omega_n^2\right\} + k}\right]\sin\frac{\pi t_{max}}{2t_o} \qquad (E_2)$$

Once t_{max} is known from (E_2), Eq. (E_1) can be used to find x_{max} as:

$$x_{max} = x(t = t_{max}) = \frac{F_o}{k}\left(1 - \cos\omega_n t_{max}\right)$$
$$+ \frac{F_o}{m\left\{\left(\frac{\pi}{2t_o}\right)^2 - \omega_n^2\right\}}\left(\cos\frac{\pi t_{max}}{2t_o} - \cos\omega_n t_{max}\right) \qquad (E_3)$$

For $t > t_0$:

$$x(t) = \frac{F_0}{k} \cos \omega_n (t - t_0)$$

$$- \sin \omega_n (t - t_0) \left\{ \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}$$

$$+ \cos \omega_n t \left\{ -\frac{F_0}{k} - \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} - \omega_n \right)} + \frac{F_0}{2m\omega_n \left(\frac{\pi}{2t_0} + \omega_n \right)} \right\}$$

For t_{max}, $\frac{dx}{dt} = 0$ $\quad (E_4)$

i.e.,

$$-\frac{F_0 \omega_n}{k} \sin \omega_n (t_{max} - t_0) - \frac{F_0 \pi \omega_n \cos \omega_n (t_{max} - t_0)}{2m\omega_n t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}}$$

$$+ \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2 \omega_n}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n t_{max} = 0 \qquad (E_5)$$

Once t_{max} is found by solving Eq. (E_5), x_{max} can be found from Eq. (E_4) as

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k} \cos \omega_n (t_{max} - t_0)$$

$$- \frac{\pi F_0}{2\omega_n m t_0 \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \sin \omega_n (t_{max} - t_0)$$

$$- \frac{F_0 \left(\frac{\pi}{2t_0} \right)^2}{k \left\{ \left(\frac{\pi}{2t_0} \right)^2 - \omega_n^2 \right\}} \cos \omega_n t_{max} \qquad (E_6)$$

Eqs. (E_3) and (E_6) can be used to plot x_{max} versus ω_n to get the displacement response spectrum.

4.37 Base acceleration $= \ddot{y}(t) = a_0 \left(1 - \sin \frac{\pi t}{2 t_0} \right)$ $\quad (E_1)$

For an undamped system, the relative displacement is given by Eq. (4.36):

$$z(t) = -\frac{1}{\omega_n} \int_0^t \ddot{y}(\tau) \sin \omega_n (t - \tau) \, d\tau$$

$$= -\frac{1}{\omega_n} \left[\int_0^t a_0 \sin \omega_n (t - \tau) \, d\tau - \int_0^t a_0 \sin \frac{\pi t}{2t_0} \sin \omega_n (t - \tau) \, d\tau \right]$$

(E_2)

Here

$$\int_0^t \sin \omega_n (t-\tau)\, d\tau = \left(\frac{1 - \cos \omega_n t}{\omega_n} \right) \qquad (E_3)$$

from Eq. (E_5) in the solution of problem 4.26.

and

$$\int_0^t \sin \frac{\pi \tau}{2 t_0} \left(\sin \omega_n t \cos \omega_n \tau - \cos \omega_n t \sin \omega_n \tau \right) d\tau$$

$$= \sin \omega_n t \left\{ - \frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t$$

$$- \cos \omega_n t \left\{ \frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)\tau}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right\}_0^t \qquad (E_4)$$

Thus the solution, Eq. (E_2), can be finally expressed as

$$z(t) = - \frac{a_0}{\omega_n} \left(1 - \frac{\cos \omega_n t}{\omega_n} \right) - \frac{a_0}{\omega_n} \left\{ \sin \omega_n t \cdot \left[\frac{\cos\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} + \frac{\cos\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} - 2 \right] + \cos \omega_n t \cdot \left[\frac{\sin\left(\frac{\pi}{2t_0} - \omega_n\right)t}{2\left(\frac{\pi}{2t_0} - \omega_n\right)} - \frac{\sin\left(\frac{\pi}{2t_0} + \omega_n\right)t}{2\left(\frac{\pi}{2t_0} + \omega_n\right)} \right] \right\} \qquad (E_5)$$

4.38 During $0 \le t \le t_0$:

$$x(t) = \frac{F_0}{k} \left(1 - \frac{t}{t_0} - \cos \omega_n t + \frac{1}{\omega_n t_0} \sin \omega_n t \right) \qquad (E_1)$$

$\dot{x}(t) = 0$ gives $\omega_n t_0 \sin \omega_n t_m = 1 - \cos \omega_n t_m$

i.e. $\omega_n t_m = 2 \tan^{-1}(\omega_n t_0) \qquad (E_2)$

(E_1) becomes

$$\frac{x_m}{(F_0/k)} = 1 - \frac{t_m}{t_0} - \cos \omega_n t_m + \frac{1}{\omega_n t_0} \sin \omega_n t_m \qquad (E_3)$$

where t_m is given by (E_2).

During $t > t_0$:

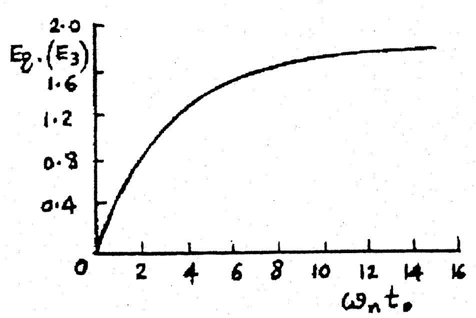
$$x(t) = \frac{F_0}{k \omega_n t_0} \left[(1 - \cos \omega_n t_0) \sin \omega_n t - (\omega_n t_0 - \sin \omega_n t_0) \cos \omega_n t \right] \qquad (E_4)$$

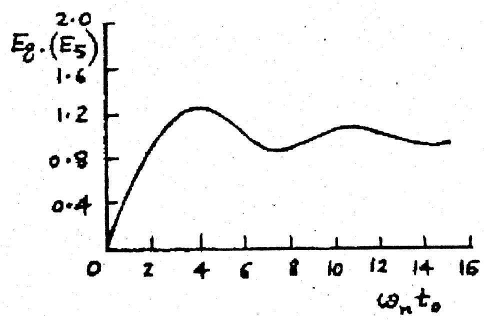
i.e. $\frac{x(t)\, k\, \omega_n t_0}{F_0} = \underset{\sim}{x}(t) = A \sin \omega_n t + B \cos \omega_n t$

where $A = 1 - \cos \omega_n t_0$; $B = -(\omega_n t_0 - \sin \omega_n t_0)$

Since $\underset{\sim}{x}\big|_{max} = \sqrt{A^2 + B^2}$,

$$\frac{x_m}{(F_0/k)} = \frac{1}{\omega_n t_0} \left[(1 - \cos \omega_n t_0)^2 + (\omega_n t_0 - \sin \omega_n t_0)^2 \right]^{1/2} \qquad (E_5)$$





(4.39) From Example 4.7, the response of the building frame is given by

$$x(t) = \frac{F_0}{k}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right], \quad 0 \le t \le t_0 \qquad (E_1)$$

and

$$x(t) = \frac{F_0}{k\,\omega_n t_0}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right], \quad t > t_0 \qquad (E_2)$$

(i) For $0 \le t \le t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in (E_1) must be maximum. This implies that

$$\frac{d}{dt}\left[1 - \frac{t}{t_0} - \cos\omega_n t + \frac{1}{\omega_n t_0}\sin\omega_n t\right] = 0$$

i.e., $\quad \omega_n t_0 \sin\omega_n t = 1 - \cos\omega_n t$

i.e., $\quad \omega_n t_0 \cos\frac{\omega_n t}{2} = \sin\frac{\omega_n t}{2}$

i.e., $\quad \tan\frac{\omega_n t}{2} = \omega_n t_0 \qquad (E_3)$

Thus, if $x(t)$ attains its maximum value at $t = t_{max}$, Eq. (E_3) gives

$$t_{max} = \frac{2}{\omega_n}\tan^{-1}(\omega_n t_0) \qquad (E_4)$$

Once t_{max} is known from (E_4), (E_1) gives

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k}\left\{1 - \frac{t_{max}}{t_0} - \cos\omega_n t_{max} + \frac{1}{\omega_n t_0}\sin\omega_n t_{max}\right\} \qquad (E_5)$$

(ii) For $t > t_0$:

For $x(t)$ to be maximum, the quantity inside square brackets in Eq. (E_2) must be maximum. This implies that

$$\frac{d}{dt}\left[(1-\cos\omega_n t_0)\sin\omega_n t - (\omega_n t_0 - \sin\omega_n t_0)\cos\omega_n t\right] = 0$$

i.e.,

$$(1-\cos\omega_n t_0)\,\omega_n\cos\omega_n t + (\omega_n t_0 - \sin\omega_n t_0)\,\omega_n \sin\omega_n t = 0$$

i.e.,

$$\tan\omega_n t = -\left(\frac{1-\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \qquad (E_6)$$

If $x(t)$ attains its maximum at $t = t_{max}$, Eq. (E_6) gives

$$t_{max} = \frac{1}{\omega_n}\tan^{-1}\left(\frac{-1+\cos\omega_n t_0}{\omega_n t_0 - \sin\omega_n t_0}\right) \qquad (E_7)$$

Once t_{max} is computed from (E_7), (E_2) gives

$$x_{max} = x(t = t_{max}) = \frac{F_0}{k\,\omega_n t_0}\left[(1-\cos\omega_n t_0)^2 + (\omega_n t_0 - \sin\omega_n t_0)^2\right]^{\frac{1}{2}} \qquad (E_8)$$

Given data:

$m = 5000$ kg, $F_0 = 4\times10^6$ N, $t_0 = 0.4$ sec, $x_{max} \le 0.01$ m.

$\omega_n = \sqrt{k/m} = 0.01414\sqrt{k}$

Procedure:

1. Assume a series of values of k.
2. Find t_{max} using Eqs. (E_4) and (E_7).
3. Find x_{max} using Eqs. (E_5) and (E_8).
4. Select k such that $x_{max} \le 0.01$ m in Eq. (E_5) or (E_8).

Sample computer program and results are shown below.

```
      XM=5000.0
      FO=4.0E+6
      TO=0.4
      XK=0.0
      DO 10 I=1,100
      XK=XK+1.0E+7
      OMN=0.01414*SQRT(XK)
      TMAX1=(2.0/OMN)*ATAN(OMN*TO)
      XMAX1=(FO/XK)*(1.0-(TMAX1/TO)-COS(OMN*TMAX1)+SIN(OMN*TMAX1)/
     2 (OMN*TO))
      XNR=-(1.0-COS(OMN*TO))
      XDR=(OMN*TO-SIN(OMN*TO))
      TMAX2=ATAN(XNR/XDR)/OMN
      X1=(1.0-COS(OMN*TO))**2
      X2=(OMN*TO-SIN(OMN*TO))**2
      X3=(X1+X2)**0.5
      XMAX2=X3*FO/(XK*OMN*TO)
      PRINT 5, I, XK, TMAX1, XMAX1, TMAX2, XMAX2
5     FORMAT(I5,2X,E15.4,2X,2E12.4,2X,2E12.4)
10    CONTINUE
      STOP
      END
```

1	0.1000E+08	0.6776E-01	0.7322E+00	-0.5134E-03	0.41[illegible]+00
2	0.2000E+08	0.4843E-01	0.3758E+00	-0.8204E-05	0.19[illegible]+00
3	0.3000E+08	0.3973E-01	0.2534E+00	-0.3859E-04	0.13[illegible]+00
4	0.4000E+08	0.3450E-01	0.1914E+00	-0.4108E-03	0.10[illegible]+00
5	0.5000E+08	0.3092E-01	0.1538E+00	-0.4234E-03	0.78[illegible]-01
⋮					

These results indicate that the required stiffness is
$k = 8 \times 10^8$ N/m.

4.40 Let d = thickness of bracket. Then, from Example 4.12, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \frac{W\,\ell^3}{3\,E\,I} = \frac{(0.5\,d + 0.4)}{d^3}\;7.9994\,(10^{-4})\text{ in}$$

We need to use a trial and error procedure to find the correct value of d.

Let d = 1 in:

$w = 0.5$ lb, $W = 0.9$ lb, $I = 0.04167$ in^4, $\delta_{st} = 0.9\,(7.9994)\,(10^{-4}) = 7.19946\,(10^{-4})$ in,

$$\tau_n = 2\,\pi\sqrt{\frac{\delta_{st}}{g}} = \sqrt{\frac{7.19946\,(10^{-4})}{386.4}} = 0.008577\text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.008577} = 11.6591$$

From Fig. 4.49, shock amplification factor (A_a) corresponding to $t_0/\tau_n = 11.6591$ is $A_a \approx 2.0$.
Dynamic load at end of cantilever = $P_d = A_a\,M\,a_s = (2.0)\,(0.9/g)\,(100g) = 180.0$ lb.

$$\sigma_{max} = \frac{M_b\,c}{I} = \frac{(180\,(10))\,(1.0/2)}{0.04167} = 21598.2721\text{ lb/in}^2$$

Since this is smaller than the permissible value of 26000 psi, we choose a smaller value of d next.

Let d = 0.9 in:

$w = 0.45$ lb, $W = 0.85$ lb, $I = 0.03038$ in^4, $\delta_{st} = \dfrac{0.85\,(7.9994\,(10^{-4}))}{0.9^3} = 9.3271\,(10^{-4})$ in

$$\tau_n = 2\,\pi\sqrt{\frac{\delta_{st}}{g}} = 2\,\pi\sqrt{\frac{9.3271\,(10^{-4})}{386.4}} = 0.009762\text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.009762} = 10.2438$$

$A_a \approx 2.0$, $P_d = A_a\,M\,a_s = (2.0)\,(0.85/g)\,(100g) = 170.0$ lb

$$\sigma_{max} = \frac{M_b\,c}{I} = \frac{(170\,(10))\,(0.9/2)}{0.03038} = 25181.0402\text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we take d = 0.9 in.

4.41 Let d = thickness of bracket. Then from Example 4.12, self weight of beam = w = 0.5 d lb, total weight at free end of beam = W = 0.5 d + 0.4 lb, moment of inertia of beam cross section = I = 0.04167 d^3 in^4, static deflection of beam under W:

$$\delta_{st} = \left(\frac{0.5\ d + 0.4}{d^3}\right) 7.9994\ (10^{-4})\ in$$

We need to use a trial and error procedure to find the correct value of d. However, since the shock amplification factor, for large values of t_0/τ_n, for the triangular pulse of Fig. 4.50 is similar to that of the pulse shown in Fig. 4.11, we start with d = 0.6 in. This gives:
w = 0.3 lb, W = 0.7 lb, I = (0.04167) (0.216) = 0.009001 in^4, $\delta_{st} = \frac{0.7}{0.6^3}\ (7.9994\ (10^{-4})) = 25.9240\ (10^{-4})$ in, $\tau_n = 2\ \pi \sqrt{\frac{25.9240\ (10^{-4})}{386.4}} =$ 0.01627 sec, t_0/τ_n = (0.1/0.01627) = 6.1445. From Fig. 4.50, we find shock amplification factor as $A_a \approx 1.1$, dynamic load at end of beam = $P_d = A_a\ M\ a_a$ = (1.1) (0.7/g) (100g) = 77.0 lb, maximum bending stress at root of beam = $\sigma_{max} = \frac{M_b\ c}{I}$ = (77(10)) (0.6/2) / (0.009001) = 25663.8151 psi. Since this stress is very close to the maximum permissible value, we select d = 0.6 in as the design.

4.42 Let d = thickness of bracket (beam). Self weight of beam = w = d (1/2) (16) (0.1) = 0.8 d lb, total load at middle of beam = W = (1 + 0.8 d) lb, area moment of inertia of beam cross section = I = $\frac{1}{12}\ (\frac{1}{2})\ d^3\ in^4$. Static deflection of beam at middle due to W:

$$\delta_{st} = \frac{W\ \ell^3}{192\ E\ I} = \frac{(1 + 0.8\ d)\ (16^3)}{192\ (10^7)\ (0.04167\ d^3)} = (1 + 0.8\ d)\ (51.1959\ (10^{-6}))\ in$$

We need to use a trial and error procedure to determine the correct value of d.

Let d = 0.4 in:

w = 0.32 lb, W = 1.32 lb, I = 0.002667 in^4, $\delta_{st} = 67.5786\ (10^{-6})$ in,

$$\tau_n = 2\ \pi \sqrt{\frac{67.5786\ (10^{-6})}{386.4}} = 0.002628\ sec$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002628} = 38.0569$$

From Fig. 4.11(b), $A_a \approx 1.1$, and the dynamic load on beam is given by $P_d = A_a\ M\ a_s$ where M = total mass of beam and a_s = acceleration due to shock = 100 g. Thus P_d = (1.1) (1.32/g) (100 g) = 145.2 lb. Maximum bending moment in a fixed-fixed beam due to load (F) at the middle is given by $M_b = \frac{F\ \ell}{8}$ so that

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{145.2\ (16)}{8}\right) (\frac{0.4}{2})}{0.002667} = 21777.2778\ lb/in^2$$

Since this stress is smaller than the maximum permissible value of 26000 psi, we next select a smaller value of d.

Let d = 0.35 in:

$w = 0.28$ lb, $W = 1.28$ lb, $I = 0.001787 \text{ in}^4$, $\delta_{st} = 65.5307\ (10^{-6})$ in,

$$\tau_n = 2\pi \sqrt{\frac{65.5307\ (10^{-6})}{386.4}} = 0.002587 \text{ sec}$$

$$\frac{t_0}{\tau_n} = \frac{0.1}{0.002587} = 38.6469$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.28/g)\ (100g) = 140.8$ lb

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{140.8\ (16)}{8}\right)\left(\frac{0.35}{2}\right)}{0.001787} = 27576.9446 \text{ lb/in}^2$$

Since this stress exceeds the permissible value, we increase the value of d.

Let d = 0.37 in:

$w = 0.296$ lb, $W = 1.296$ lb, $I = 0.002111 \text{ in}^4$, $\delta_{st} = 66.3499\ (10^{-6})$ in

$$\tau_n = 2\pi \sqrt{\frac{66.3499\ (10^{-6})}{386.4}} = 0.002604 \text{ sec}$$

$$\frac{t_0}{\tau_n} = 38.4024$$

From Fig. 4.11(b), $A_a \approx 1.1$, $P_d = (1.1)\ (1.296/g)\ (100\ g) = 142.56$ lb, and

$$\sigma_{max} = \frac{M_b\ c}{I} = \frac{\left(\frac{142.56\ (16)}{8}\right)\left(\frac{0.37}{2}\right)}{0.002111} = 24986.8309 \text{ lb/in}^2$$

Since this stress is close to the maximum permissible value, we select the design as d = 0.37 in.

4.43 $W = m\ g = 100000$ lb, $\zeta = 0.05$, $\sigma_y = 30000$ psi,

σ_{max} = maximum permissible stress $= \frac{\sigma_y}{2} = \frac{30000}{2} = 15000$ psi,

$$\tau_n = \frac{2\pi}{\omega_n} = 2\pi\sqrt{\frac{m}{k}} = 2\pi\sqrt{\frac{100000}{386.4\ k}} = \frac{101.0793}{\sqrt{k}} \text{ sec}$$

We need to use a trial and error procedure to find k.

Let k = 10000 lb/in:

$$k = 10^4 = \frac{3\,E\,I}{\ell^3} = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 24000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\,(0.5904) \text{ with } \frac{d_i}{d_o} = 0.8$$

$$d_o^4 = 82.8121\,(10^4) \text{ in}^4$$

$$d_o = 30.1664 \text{ in} \; ; \; d_i = 24.1331 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{10^4}} \approx 1 \text{ sec}$$

From Fig. 4.14, for $\tau_n = 1$ sec and $\varsigma = 0.05$, we find $S_v = 25$ in/sec, $S_d = 4.2$ in and $S_a = 0.42$ g.

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{100000}{g}\,(0.42g) = 42000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (42000)\,(50\,(12)) = 25.2\,(10^6) \text{ lb–in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(25.2\,(10^6))\,(30.1664/2)}{24\,(10^3)} = 15837.36 \text{ lb/in}^2$$

Since this stress is slightly smaller than the maximum permissible value, we choose a larger value of k.

Let k = 20000 lb/in:

$$k = 2\,(10^4) = \frac{3\,(30\,(10^6))\,I}{600^3}$$

$$I = 48000 \text{ in}^4 = \frac{\pi}{64}\, d_o^4\,(0.5904)$$

$$d_o^4 = 165.6243\,(10^4) \text{ in}^4$$

$$d_o = 35.8741 \text{ in} \; ; \; d_i = 28.6993 \text{ in}$$

$$\tau_n = \frac{101.0793}{\sqrt{20000}} = 0.7147 \text{ sec}$$

From Fig. 4.14, we find

$$S_v = 26 \text{ in/sec}, \; S_d = 3 \text{ in}, \; S_a = 0.6 \text{ g}$$

Maximum shear force in column:

$$F_{max} = \frac{W}{g}\, S_a = \frac{10^5}{g}\,(0.6g) = 60000 \text{ lb}$$

Maximum bending moment:

$$M_b = F_{max}\, h = (60000)\,(600) = 36\,(10^6) \text{ lb–in}$$

Maximum bending stress:

$$\sigma_b = \frac{M_b\, c}{I} = \frac{(36\,(10^6))\,(35.8741/2)}{48\,(10^3)} = 13452.7875 \text{ lb/in}^2$$

Since this stress is less than the maximum permissible value, we choose the inner and outer diameters of the column as $d_i = 28.6993$ in and $d_o = 35.8741$ in.

4.44 m g = 5000 lb, $\varsigma = 0.02$. From Fig. 4.15, in order to have $S_a \approx 1$ g, we need to have $\tau_n = 0.2$ sec. Thus

$$\tau_n = 0.2 = \frac{2\pi}{\omega_n} \quad ; \quad \omega_n = 31.416 \text{ rad/sec} = \sqrt{\frac{k}{m}}$$

$$k = (31.416)^2 \, m = (31.416^2)\,(5000/386.4) = 12771.2870 \text{ lb/in}$$

4.45 Equation of motion is $\ddot{x} + \omega_n^2 x = \frac{F_0}{m} e^{i\omega t}$

For zero initial conditions

$$(s^2 + \omega_n^2)\,\bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{s - i\omega} \quad ; \quad \bar{x}(s) = \frac{F_0}{m} \cdot \frac{1}{(s^2+\omega_n^2)} \cdot \frac{s+i\omega}{(s^2+\omega^2)}$$

Inverse Laplace transformation gives

$$x(t) = \frac{F_0}{m} \frac{1}{(\omega_n^2 - \omega^2)} \left\{ e^{i\omega t} - \left(\cos \omega_n t + \frac{i\omega}{\omega_n} \sin \omega_n t \right) \right\}$$

The terms containing $\cos \omega_n t$ and $\sin \omega_n t$ denote the transient part of the response and hence can be neglected. Thus the steady state response can be expressed as

$$x(t) = \frac{F_0}{k} \left(\frac{1}{1-r^2} \right) e^{i\omega t} \qquad \text{where } r = \omega/\omega_n .$$

4.46 $\bar{F}(s) = \frac{F_0}{s}$

From Eq. (4.57),

$$\bar{x}(s) = \frac{F_0}{m s\,(s^2 + 2\zeta\omega_n s + \omega_n^2)} + \left(\frac{s + 2\zeta\omega_n}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) x_0 + \left(\frac{1}{s^2 + 2\zeta\omega_n s + \omega_n^2} \right) \dot{x}_0$$

Inverse transformation gives

$$x(t) = \frac{F_0}{m} \left\{ 1 - \frac{e^{-\zeta\omega_n t}}{\sqrt{1-\zeta^2}} \sin(\omega_d t + \phi_1) \right\} + \frac{x_0}{\sqrt{1-\zeta^2}} e^{-\zeta\omega_n t} \sin(\omega_d t + \phi_1) + \frac{\dot{x}_0}{\omega_d} e^{-\zeta\omega_n t} \sin \omega_d t$$

where $\phi_1 = \cos^{-1}(\zeta)$.

4.47 The forcing function can be expressed as

$$F(t) = F_0 \{ u(t) - u(t - t_0) \}$$

where $u(t-\tau)$ is the unit step function applied at $t = \tau$. The Laplace transform of $F(t)$ is

$$\bar{F}(s) = F_0 \left(\frac{1}{s} - \frac{1}{s} e^{-s t_0} \right)$$

The equation of motion $m\ddot{x} + kx = F(t)$ gives

$$\bar{x}(s) = \frac{F_0}{s}\left(1 - e^{-st_0}\right)\frac{1}{ms^2+k} = \frac{F_0}{m}\left\{\frac{1}{s(s^2+\omega_n^2)} - \frac{e^{-st_0}}{s(s^2+\omega_n^2)}\right\}$$

Since $\mathcal{L}^{-1}\left\{\frac{1}{s(s^2+a^2)}\right\} = \frac{1}{a^2}(1-\cos at)$,

$$x(t) = \frac{F_0}{m\omega_n^2}(1-\cos\omega_n t)\cdot u(t) - \frac{F_0}{m\omega_n^2}\{1-\cos\omega_n(t-t_0)\}\, u(t-t_0)$$

Hence

$$x(t) = \begin{cases} \dfrac{F_0}{m\omega_n^2}(1-\cos\omega_n t) & \text{for } 0 \le t \le t_0 \\[2ex] \dfrac{F_0}{m\omega_n^2}\{\cos\omega_n(t-t_0) - \cos\omega_n t\} & \text{for } t \ge t_0 \end{cases}$$

4.48

$$\dot{x}(t) = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\Big\{-(-\zeta\omega_n)e^{-\zeta\omega_n(t-t_i)}\cos\omega_d(t-t_i)$$

$$+ e^{-\zeta\omega_n(t-t_i)}\omega_d\sin\omega_d(t-t_i) - (-\zeta\omega_n)e^{-\zeta\omega_n(t-t_i)}\frac{\zeta\omega_n}{\omega_d}\sin\omega_d(t-t_i)$$

$$- e^{-\zeta\omega_n(t-t_i)}\frac{\zeta\omega_n}{\omega_d}\cdot\omega_d\cdot\cos\omega_d(t-t_i)\Big\}$$

$$= \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left\{e^{-\zeta\omega_n(t-t_i)}\right\}\left(\omega_d + \frac{\zeta^2\omega_n^2}{\omega_d}\right)\sin\omega_d(t-t_i)$$

$$\dot{x}_j = \dot{x}(t=t_j) = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left(\omega_d + \frac{\zeta^2\omega_n^2}{\omega_d}\right)e^{-\zeta\omega_n(t_j-t_i)}\sin\omega_d(t_j-t_i)$$

4.49 Eq. (4.68): Differentiating Eq. (4.66),

$$\dot{x}(t) = \frac{F_j}{k}e^{-\zeta\omega_n(t-t_{j-1})}\omega_d\left(1+\frac{\zeta^2\omega_n^2}{\omega_d^2}\right)\sin\omega_d(t-t_{j-1}) + \omega_d\, e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\Big\{-x_{j-1}\sin\omega_d(t-t_{j-1}) + \frac{\dot{x}_{j-1}}{\omega_d}\cos\omega_d(t-t_{j-1})$$

$$- \frac{\zeta\omega_n}{\omega_d^2}(\dot{x}_{j-1} + \zeta\omega_n x_{j-1})\sin\omega_d(t-t_{j-1})\Big\}$$

For $t = t_j$,

$$\dot{x}_j = \frac{F_j}{k}e^{-\zeta\omega_n\Delta t_j}\omega_d\left(1+\frac{\zeta^2\omega_n^2}{\omega_d^2}\right)\sin\omega_d\Delta t_j + \omega_d\, e^{-\zeta\omega_n\Delta t_j}\Big\{-x_{j-1}\sin\omega_d\Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_d}\cos\omega_d\Delta t_j - \frac{\zeta\omega_n}{\omega_d^2}(\dot{x}_{j-1} + \zeta\omega_n x_{j-1})\sin\omega_d\Delta t_j\Big\}$$

Eq. (4.71):

$$\dot{x}(t) = \frac{\Delta F_j}{k\,\Delta t_j}\Big[1 - \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})}\frac{2\zeta}{\omega_n}\cos\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\frac{2\zeta}{\omega_n}\cdot\omega_d\cdot\sin\omega_d(t-t_{j-1}) + \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\omega_d^2-\zeta^2\omega_n^2}{\omega_n^2\omega_d}\right)\sin\omega_d(t-t_{j-1})$$

$$- e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\omega_d^2-\zeta^2\omega_n^2}{\omega_n^2\,\omega_d}\right)\omega_d\cos\omega_d(t-t_{j-1})\Big] + \frac{F_{j-1}}{k}\Big[\zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\cos\omega_d(t-t_{j-1}) + e^{-\zeta\omega_n(t-t_{j-1})}\omega_d\sin\omega_d(t-t_{j-1}) + \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})} *$$

$$\left(\frac{\zeta\omega_n}{\omega_d}\right)\sin\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\zeta\omega_n}{\omega_d}\right)\omega_d\cdot\cos\omega_d(t-t_{j-1})\Big]$$

$$-\zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})} x_{j-1}\cos\omega_d(t-t_{j-1}) - e^{-\zeta\omega_n(t-t_{j-1})} x_{j-1}\omega_d *$$

$$\sin\omega_d(t-t_{j-1}) - \zeta\omega_n e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\dot{x}_{j-1}+\zeta\omega_n x_{j-1}}{\omega_d}\right)\sin\omega_d(t-t_{j-1})$$

$$+ e^{-\zeta\omega_n(t-t_{j-1})}\left(\frac{\dot{x}_{j-1}+\zeta\omega_n x_{j-1}}{\omega_d}\right)\omega_d\cos\omega_d(t-t_{j-1})$$

Simplification of this equation gives at $t = t_j$,

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left[1 - e^{-\zeta\omega_n\Delta t_j}\left\{\cos\omega_d\Delta t_j + \frac{\zeta\omega_n}{\omega_d}\sin\omega_d\Delta t_j\right\}\right]$$

$$+ \frac{F_{j-1}}{k}\, e^{-\zeta\omega_n\Delta t_j}\,\frac{\omega_n^2}{\omega_d}\sin\omega_d\Delta t_j$$

$$+ e^{-\zeta\omega_n\Delta t_j}\left\{\dot{x}_{j-1}\cos\omega_d\Delta t_j - \frac{\zeta\omega_n}{\omega_d}\left(\dot{x}_{j-1} + \frac{\omega_n}{\zeta}x_{j-1}\right)\sin\omega_d\Delta t_j\right\}$$

4.50

i	t_i	$\dot{x}_i$, Eq. (4.68)	$\dot{x}_i$, Eq. (4.71)
1	0	0	0
2	0.1π	0.276010E+00	0.275640E+00
3	0.2π	0.462605E+00	0.461687E+00
4	0.3π	0.549249E+00	0.547683E+00
5	0.4π	0.536630E+00	0.534405E+00
6	0.5π	0.435845E+00	0.433036E+00
7	0.6π	0.266613E+00	0.263378E+00
8	0.7π	0.547668E-01	0.513272E-01
9	0.8π	-0.170711E+00	-0.174093E+00
10	0.9π	-0.380784E+00	-0.383830E+00
11	π	-0.549289E+00	-0.551730E+00

4.51 Method 1:

$$x_j = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\left\{1-\cos\omega_n(t_j-t_i)\right\}$$

$$\dot{x}_j = \frac{1}{k}\sum_{i=1}^{j-1}\Delta F_i\,\omega_n\sin\omega_n(t_j-t_i)$$

Method 2:

$$x_j = \frac{F_j}{k}\left[1-\cos\omega_n\Delta t_j\right] + x_{j-1}\cos\omega_n\Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n}\sin\omega_n\Delta t_j$$

$$\dot{x}_j = \frac{F_j\,\omega_n}{k}\sin\omega_n\Delta t_j + \omega_n\left\{-x_{j-1}\sin\omega_n\Delta t_j + \frac{\dot{x}_{j-1}}{\omega_n}\cos\omega_n\Delta t_j\right\}$$

Method 3:

$$x_j = \frac{\Delta F_j}{k\,\Delta t_j}\left\{\Delta t_j - \frac{1}{\omega_n}\sin\omega_n\Delta t_j\right\} - \frac{F_{j-1}}{k}\left(1-\cos\omega_n\Delta t_j\right) + x_{j-1}\cos\omega_n\Delta t_j$$

$$+ \frac{\dot{x}_{j-1}}{\omega_n}\sin\omega_n\Delta t_j$$

$$\dot{x}_j = \frac{\Delta F_j}{k\,\Delta t_j}\left(1-\cos\omega_n \Delta t_j\right) + \frac{F_{j-1}}{k}\,\omega_n \sin\omega_n \Delta t_j + \dot{x}_{j-1}\cos\omega_n \Delta t_j - \omega_n x_{j-1}\sin\omega_n \Delta t_j$$

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.489135E-01	0.412870E-01	0.451034E-01	0.463829E-01
3	0.183327E+00	0.153639E+00	0.168413E+00	0.170863E+00
4	0.374871E+00	0.311494E+00	0.342969E+00	0.346380E+00
5	0.590262E+00	0.485757E+00	0.537537E+00	0.541621E+00
6	0.794774E+00	0.646981E+00	0.720014E+00	0.724433E+00
7	0.955998E+00	0.768556E+00	0.860896E+00	0.865287E+00
8	0.104732E+01	0.829582E+00	0.936446E+00	0.940461E+00
9	0.105081E+01	0.817133E+00	0.931268E+00	0.934605E+00
10	0.959172E+00	0.727695E+00	0.840009E+00	0.842438E+00
11	0.776638E+00	0.566437E+00	0.668027E+00	0.669410E+00
I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.309017E+00	0.260676E+00	0.284772E+00	0.284646E+00
3	0.539444E+00	0.448685E+00	0.493775E+00	0.493286E+00
4	0.669917E+00	0.547974E+00	0.608327E+00	0.607283E+00
5	0.690013E+00	0.552279E+00	0.620126E+00	0.618406E+00
6	0.601222E+00	0.465650E+00	0.531993E+00	0.529561E+00
7	0.416706E+00	0.301948E+00	0.357498E+00	0.354412E+00
8	0.159909E+00	0.833536E-01	0.119506E+00	0.115921E+00
9	-0.137878E+00	-0.161957E+00	-0.152198E+00	-0.156045E+00
10	-0.440725E+00	-0.402734E+00	-0.423989E+00	-0.427799E+00
11	-0.711751E+00	-0.615408E+00	-0.661862E+00	-0.665302E+00

4.52 $\omega_n = \sqrt{k/m} = \sqrt{50/2} = 5$ rad/s; $\zeta = \frac{c}{2m\omega_n} = \frac{2}{2(2)(5)} = 0.1$

The problem-dependent data to be used in Program 5 and the output are given.

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(11),FF(11),DELF(11),T(11),X(11),XD(11),X1(11),
     2 XD1(11),X2(11),XD2(11),X3(11),XD3(11),X4(11),XD4(11)
      DATA F/-8.,-12.,-15.,-13.,-11.,-7.,-4.,3.,10.,15.,18./
      DO 5 I=1,11
    5 FF(I)=F(I)
      XAI=0.1
      OMN=5.0
      DELT=0.1
      XK=50.0
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)

2	-0.947628E-02	-0.355361E-01	-0.284289E-01	-0.308375E-01
3	-0.415895E-01	-0.124570E+00	-0.110554E+00	-0.117557E+00
4	-0.888374E-01	-0.231948E+00	-0.224265E+00	-0.229133E+00
5	-0.129452E+00	-0.316847E+00	-0.330743E+00	-0.326279E+00
6	-0.140543E+00	-0.345356E+00	-0.391810E+00	-0.371424E+00
7	-0.106067E+00	-0.292742E+00	-0.381288E+00	-0.341045E+00
8	-0.149353E-01	-0.146366E+00	-0.284987E+00	-0.219841E+00
9	0.138845E+00	0.786267E-01	-0.100887E+00	-0.136816E-01
10	0.341364E+00	0.340894E+00	0.147643E+00	0.244535E+00
11	0.559359E+00	0.589762E+00	0.415128E+00	0.488339E+00

I	XD(I)	XD(I)	XD(I)	XD(I)
2	-0.363160E-01	-0.136185E+00	-0.547484E+00	-0.618556E+00
3	-0.879517E-01	-0.209522E+00	-0.105218E+01	-0.105547E+01
4	-0.960236E-01	-0.208906E+00	-0.117245E+01	-0.111170E+01
5	-0.627967E-01	-0.123358E+00	-0.916858E+00	-0.762976E+00
6	0.182155E-01	0.100267E-01	-0.289472E+00	-0.106971E+00
7	0.114511E+00	0.191935E+00	0.482526E+00	0.746090E+00
8	0.238516E+00	0.375367E+00	0.138800E+01	0.167033E+01
9	0.358701E+00	0.499280E+00	0.220328E+01	0.239624E+01
10	0.429269E+00	0.522310E+00	0.265575E+01	0.268039E+01
11	0.420343E+00	0.448691E+00	0.258326E+01	0.195527E+01

4.53 The problem-dependent data to be used in Program 5 and results are given.

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51),
     2 XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
      DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
     2 20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,0.,
     3 0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
     4 0./
      DO 5 I=1,51
    5 FF(I)=F(I)
      XAI=0.0
      OMN=27.3861
      DELT=0.01
      XK=1500.0
      NP=51
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF I	METHOD #1 (FIG.4.18) X(I)	METHOD #1 (FIG.4.19) X(I)	METHOD #2 (FIG.4.20) X(I)	METHOD #3 (FIG.4.21) X(I)
2	0.496882E-04	0.993764E-04	0.496882E-04	0.662924E-04
3	0.244738E-03	0.439787E-03	0.244738E-03	0.326397E-03
4	0.669987E-03	0.109524E-02	0.669987E-03	0.860303E-03
5	0.139312E-02	0.211625E-02	0.139312E-02	0.172759E-02
6	0.245961E-02	0.352610E-02	0.245961E-02	0.296301E-02
7	0.388935E-02	0.531910E-02	0.388935E-02	0.457384E-02
8	0.567516E-02	0.746097E-02	0.567516E-02	0.653940E-02
9	0.778331E-02	0.989115E-02	0.778330E-02	0.881258E-02
10	0.101560E-01	0.125288E-01	0.101560E-01	0.113233E-01

⋮

45	0.798152E-02	0.845167E-02	0.798155E-02	0.827565E-02
46	0.831680E-02	0.807062E-02	0.831682E-02	0.825346E-02
47	0.803221E-02	0.708804E-02	0.803222E-02	0.761612E-02
48	0.714897E-02	0.557718E-02	0.714896E-02	0.641114E-02
49	0.573290E-02	0.365064E-02	0.573288E-02	0.472831E-02
50	0.388954E-02	0.145202E-02	0.388951E-02	0.269308E-02
51	0.175629E-02	-0.854840E-03	0.175624E-02	0.457121E-03

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.360601E-03	0.721201E-03	0.987545E-02	0.148443E-01
3	0.105493E-02	0.174925E-02	0.288903E-01	0.385198E-01
4	0.203123E-02	0.300753E-02	0.556274E-01	0.692620E-01
5	0.321673E-02	0.440224E-02	0.880938E-01	0.104780E+00
6	0.452309E-02	0.582945E-02	0.123870E+00	0.142425E+00
7	0.585294E-02	0.718278E-02	0.160289E+00	0.179393E+00
8	0.710715E-02	0.836136E-02	0.194637E+00	0.212929E+00
9	0.819225E-02	0.927735E-02	0.224354E+00	0.240531E+00
10	0.902736E-02	0.986248E-02	0.247224E+00	0.260144E+00
⋮				
45	0.233952E-02	-0.244374E-03	0.640694E-01	0.289820E-01
46	0.937302E-04	-0.252103E-02	0.256574E-02	-0.333924E-01
47	-0.215904E-02	-0.460977E-02	-0.591291E-01	-0.932780E-01
48	-0.425090E-02	-0.635495E-02	-0.116417E+00	-0.146211E+00
49	-0.602593E-02	-0.762648E-02	-0.165028E+00	-0.188247E+00
50	-0.735183E-02	-0.832959E-02	-0.201339E+00	-0.216253E+00
51	-0.812978E-02	-0.841188E-02	-0.222644E+00	-0.228140E+00

(4.54) *The problem-dependent data to be used in Program 5 and output are given below.*

```
C FOLLOWING 11 LINES CONTAIN PROBLEM-DEPENDENT DATA
      DIMENSION F(51),FF(51),DELF(51),T(51),X(51),XD(51),X1(51),
     2 XD1(51),X2(51),XD2(51),X3(51),XD3(51),X4(51),XD4(51)
      DATA F/0.,2.,4.,6.,8.,10.,12.,14.,16.,18.,20.,20.,20.,20.,
     2 20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,20.,0.,0.,0.,0.,
     3 0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,0.,
     4 0./
      DO 5 I=1,51
    5 FF(I)=F(I)
      XAI=0.0912872
      OMN=27.3861
      DELT=0.01
      XK=1500.0
      NP=51
C END OF PROBLEM-DEPENDENT DATA
```

VALUE OF	METHOD #1 (FIG.4.18)	METHOD #1 (FIG.4.19)	METHOD #2 (FIG.4.20)	METHOD #3 (FIG.4.21)
I	X(I)	X(I)	X(I)	X(I)
2	0.488713E-04	0.977126E-04	0.488713E-04	0.652702E-04
3	0.237610E-03	0.426350E-03	0.237610E-03	0.316883E-03
4	0.642607E-03	0.104760E-02	0.642607E-03	0.824391E-03
5	0.132081E-02	0.199900E-02	0.132080E-02	0.163535E-02
6	0.230651E-02	0.329221E-02	0.230651E-02	0.277294E-02
7	0.360999E-02	0.491347E-02	0.360999E-02	0.423546E-02
8	0.521792E-02	0.682586E-02	0.521792E-02	0.599767E-02
9	0.709550E-02	0.897307E-02	0.709550E-02	0.801378E-02
10	0.919000E-02	0.112845E-01	0.919000E-02	0.102218E-01
⋮				

45	0.440657E-02	0.534263E-02	0.440660E-02	0.491147E-02
46	0.529242E-02	0.576879E-02	0.529244E-02	0.556998E-02
47	0.575031E-02	0.575479E-02	0.575032E-02	0.579144E-02
48	0.576783E-02	0.532398E-02	0.576783E-02	0.558107E-02
49	0.536517E-02	0.452545E-02	0.536516E-02	0.497521E-02
50	0.459210E-02	0.343772E-02	0.459208E-02	0.403720E-02
51	0.352290E-02	0.215312E-02	0.352286E-02	0.285144E-02

I	XD(I)	XD(I)	XD(I)	XD(I)
2	0.350266E-03	0.700531E-03	0.963263E-02	0.145198E-01
3	0.100825E-02	0.166623E-02	0.277276E-01	0.369691E-01
4	0.191111E-02	0.281397E-02	0.525573E-01	0.653292E-01
5	0.298127E-02	0.405142E-02	0.819876E-01	0.972501E-01
6	0.413276E-02	0.528425E-02	0.113655E+00	0.130237E+00
7	0.527791E-02	0.642306E-02	0.145147E+00	0.161841E+00
8	0.633377E-02	0.738963E-02	0.174185E+00	0.189831E+00
9	0.722794E-02	0.812211E-02	0.198775E+00	0.212348E+00
10	0.790330E-02	0.857866E-02	0.217348E+00	0.228023E+00
⋮				
45	0.395372E-02	0.234794E-02	0.108730E+00	0.870512E-01
46	0.246078E-02	0.745291E-03	0.676726E-01	0.441850E-01
47	0.861748E-03	-0.833375E-03	0.236977E-01	0.197140E-03
48	-0.721946E-03	-0.227447E-02	-0.198553E-01	-0.416597E-01
49	-0.217591E-02	-0.347992E-02	-0.598408E-01	-0.784465E-01
50	-0.310079E-02	-0.437360E-02	-0.935257E-01	-0.107736E+00
51	-0.431663E-02	-0.490573E-02	-0.118769E+00	-0.127765E+00

4.55 $x(t) = \frac{1}{m\omega_d}\int_0^t F(\tau)\, e^{-\gamma\omega_n(t-\tau)} \sin\omega_d(t-\tau)\, d\tau$

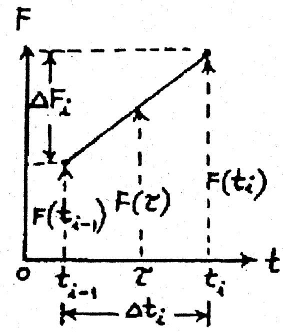
But $\sin\omega_d(t-\tau) = \sin\omega_d t\,\cos\omega_d\tau - \cos\omega_d t\,\sin\omega_d\tau$

$x(t) = \{A(t)\cdot\sin\omega_d t - B(t)\cdot\cos\omega_d t\}\,\frac{e^{-\gamma\omega_n t}}{m\omega_d}$

where

$A(t) = \int_0^t F(\tau)\, e^{\gamma\omega_n\tau}\cos\omega_d\tau\,.d\tau$

$B(t) = \int_0^t F(\tau)\, e^{\gamma\omega_n\tau}\sin\omega_d\tau\,.d\tau$



Let $F(\tau)$ be taken as a piecewise linear function during (t_{i-1}, t_i) as

$F(\tau) = F_{i-1} + \left(\frac{\tau - t_{i-1}}{t_i - t_{i-1}}\right)(F_i - F_{i-1})$

we can write

$$A(t_i) = A(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i}\,\Delta F_i\right) e^{\zeta\omega_n\tau} \cos\omega_d\tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) e^{\zeta\omega_n\tau} \cos\omega_d\tau \cdot d\tau$$

$$= A(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_1 + \left(F_{i-1} - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) P_2$$

where

$$P_1 = \int_{t_{i-1}}^{t_i} \tau\, e^{\zeta\omega_n\tau} \cos\omega_d\tau \cdot d\tau$$

$$P_2 = \int_{t_{i-1}}^{t_i} e^{\zeta\omega_n\tau} \cos\omega_d\tau \cdot d\tau$$

$$B(t_i) = B(t_{i-1}) + \int_{t_{i-1}}^{t_i} \left(\frac{\tau}{\Delta t_i}\,\Delta F_i\right) e^{\zeta\omega_n\tau} \sin\omega_d\tau \cdot d\tau$$

$$+ \int_{t_{i-1}}^{t_i} \left(F_{i-1} - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) e^{\zeta\omega_n\tau} \sin\omega_d\tau \cdot d\tau$$

$$= B(t_{i-1}) + \frac{\Delta F_i}{\Delta t_i} P_4 + \left(F_{i-1} - t_{i-1}\frac{\Delta F_i}{\Delta t_i}\right) P_3$$

where

$$P_3 = \int_{t_{i-1}}^{t_i} e^{\zeta\omega_n\tau} \sin\omega_d\tau \cdot d\tau$$

$$P_4 = \int_{t_{i-1}}^{t_i} \tau\, e^{\zeta\omega_n\tau} \sin\omega_d\tau \cdot d\tau$$

Assuming $A(t_1=0) = B(t_1=0) = 0$,

$$x(t_i) = \frac{e^{-\zeta\omega_n t_i}}{m\,\omega_d}\left[A(t_i)\sin\omega_d t_i - B(t_i)\cos\omega_d t_i\right]$$

The integrals in P_1, P_2, P_3 and P_4 can be evaluated in closed form.

The computer program and output are given.

```
C ==========================================================================
C
C PROBLEM 4.55
C NUMERICAL INTEGRATION OF DUHAMEL INTEGRAL
C
C ==========================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(21),X(21),DELT(21),T(21),A(21),B(21)
      NP=21
      XAI=0.1
      OMN=1.0
```

```
      XM=1.0
      DATA F/1.0,.8436,.6910,.5460,.4122,.2929,.1910,.1090,
     2 .04894,.01231,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0,.0/
      DO 10 I=1,21
 10   DELT(I)=0.31416
C END OF PROBLEM-DEPENDENT DATA
      A(1)=0.0
      B(1)=0.0
      OMD=OMN*SQRT(1.0-XAI**2)
      T(1)=0.0
      DO 20 I=2,NP
      T(I)=T(I-1)+DELT(I)
      TIME=T(I)
      CALL PI1(TIME,XAI,OMN,OMD,PP1)
      CALL PI2(TIME,XAI,OMN,OMD,PP2)
      CALL PI3(TIME,XAI,OMN,OMD,PP3)
      CALL PI4(TIME,XAI,OMN,OMD,PP4)
      TIME=T(I-1)
      CALL PI1(TIME,XAI,OMN,OMD,PM1)
      CALL PI2(TIME,XAI,OMN,OMD,PM2)
      CALL PI3(TIME,XAI,OMN,OMD,PM3)
      CALL PI4(TIME,XAI,OMN,OMD,PM4)
      P1=PP1-PM1
      P2=PP2-PM2
      P3=PP3-PM3
      P4=PP4-PM4
      DELF=F(I)-F(I-1)
      A(I)=A(I-1)+(DELF/DELT(I))*P1+(F(I-1)-T(I-1)*DELF/DELT(I))*P2
      B(I)=B(I-1)+(DELF/DELT(I))*P4+(F(I-1)-T(I-1)*DELF/DELT(I))*P3
      X(I)=(EXP(-XAI*OMN*T(I))/(XM*OMD))*(A(I)*SIN(OMD*T(I))-
     2 B(I)*COS(OMD*T(I)))
 20   CONTINUE
      PRINT 30
 30   FORMAT (//,2X,41H NUMERICAL EVALUATION OF DUHAMEL INTEGRAL,
     2 //,5X,2H I,6X,5H T(I),10X,5H F(I),10X,5H X(I),/)
      DO 40 I=2,NP
 40   PRINT 50, I,T(I),F(I),X(I)
 50   FORMAT (2X,I5,3E15.8)
      STOP
      END
C =================================================================
C
C SUBROUTINE PI1
C
C =================================================================
      SUBROUTINE PI1 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*COS(OMD*T)
     3 +2.0*XAI*OMN*OMD*SIN(OMD*T))
      RETURN
      END
```

```
C ==================================================================
C
C SUBROUTINE PI2
C
C ==================================================================
      SUBROUTINE PI2 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*COS(OMD*T)+OMD*SIN(OMD*T))
      RETURN
      END
C ==================================================================
C
C SUBROUTINE PI3
C
C ==================================================================
      SUBROUTINE PI3 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
      RETURN
      END
C ==================================================================
C
C SUBROUTINE PI4
C
C ==================================================================
      SUBROUTINE PI4 (T,XAI,OMN,OMD,P)
      DEN=(XAI*OMN)**2+OMD**2
      P=(T*EXP(XAI*OMN*T)/DEN)*(XAI*OMN*SIN(OMD*T)-OMD*COS(OMD*T))
     2 -(EXP(XAI*OMN*T)/(DEN**2))*(((XAI*OMN)**2-OMD**2)*SIN(OMD*T)
     3 -2.0*XAI*OMN*OMD*COS(OMD*T))
      RETURN
      END
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.31415999E+00	0.84359998E+00	0.45415938E-01
3	0.62831998E+00	0.69099998E+00	0.16377741E+00
4	0.94247997E+00	0.54600000E+00	0.32499743E+00
5	0.12566404E+01	0.41219997E+00	0.49747676E+00
6	0.15708008E+01	0.29290003E+00	0.65152758E+00
7	0.18849611E+01	0.19099998E+00	0.76239210E+00
8	0.21991215E+01	0.10900003E+00	0.81256396E+00
9	0.25132818E+01	0.48939999E-01	0.79324198E+00
10	0.28274422E+01	0.12309998E-01	0.70482814E+00
11	0.31416025E+01	0.00000000E+00	0.55646867E+00
12	0.34557629E+01	0.00000000E+00	0.36454141E+00
13	0.37699232E+01	0.00000000E+00	0.14971566E+00
14	0.40840836E+01	0.00000000E+00	-0.66231847E-01
15	0.43982439E+01	0.00000000E+00	-0.26274461E+00
16	0.47124043E+01	0.00000000E+00	-0.42236131E+00
17	0.50265646E+01	0.00000000E+00	-0.53218621E+00
18	0.53407249E+01	0.00000000E+00	-0.58483124E+00
19	0.56548853E+01	0.00000000E+00	-0.57878375E+00
20	0.59690456E+01	0.00000000E+00	-0.51819199E+00
21	0.62832060E+01	0.00000000E+00	-0.41212624E+00

(4.56) The computer program of problem 4.55 can be used to find the relative displacement $z(t)$ of the water tank provided $-m\ddot{y}(\tau)$ is used in place of $F(\tau)$.

Here $\zeta = 0$, $\omega_n = 22.3607$ rad/sec and

$$F(\tau) = -10000 \times 9.81 \times \ddot{y}(\tau) \quad \text{if } \ddot{y} \text{ is in } g\text{'s}.$$

The problem-dependent data for the program of problem 4.55 and the output are given below.

```
C ================================================================
C
C PROBLEM 4.56
C
C ================================================================
C PROBLEM-DEPENDENT DATA
      DIMENSION F(15),X(15),DELT(15),T(15),A(15),B(15)
      NP=15
      XAI=0.0
      OMN=22.3607
      XM=10000.0
      DATA F/.0,.45,-.8,-.9,-.6,-.75,-.7,.55,1.75,1.65,.25,
     2 -1.1,-1.4,-1.05,.0/
      DO 10 I=1,15
   10 DELT(I)=0.025
      DO 11 I=1,15
   11 F(I)=-XM*9.81*F(I)
C END OF PROBLEM-DEPENDENT DATA
```

```
NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

    I        T(I)              F(I)              X(I)

    2 0.24999999E-01-0.44144996E+05-0.45271125E-03
    3 0.49999997E-01 0.78480000E+05-0.17453337E-02
    4 0.74999988E-01 0.88290000E+05 0.11150776E-02
    5 0.99999964E-01 0.58860004E+05 0.86095147E-02
    6 0.12499994E+00 0.73575000E+05 0.17519452E-01
    7 0.14999992E+00 0.68670000E+05 0.25374368E-01
    8 0.17499989E+00-0.53955000E+05 0.28478131E-01
    9 0.19999987E+00-0.17167500E+06 0.19676924E-01
   10 0.22499985E+00-0.16186494E+06-0.42601228E-02
   11 0.24999982E+00-0.24525000E+05-0.35448216E-01
   12 0.27499980E+00 0.10791006E+06-0.57387743E-01
   13 0.29999977E+00 0.13734000E+06-0.56341648E-01
   14 0.32499975E+00 0.10300506E+06-0.30433871E-01
   15 0.34999973E+00 0.00000000E+00 0.10307007E-01
```

4.57 The problem-dependent data (to be used in the program of Problem 4.55) and output are given.

```
C PROBLEM-DEPENDENT DATA
      DIMENSION F(30),X(30),DELT(30),T(30),A(30),B(30)
      NP=30
      XAI=0.0
      OMN=8.660254
      XM=2.0
      DATA F /60.0,60.0,60.0,60.0,60.0,100.0,100.0,100.0,100.0,100.0,
     2 30.0,30.0,30.0,30.0,30.0,30.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,0.0,
     3 0.0,0.0,0.0,0.0,0.0,0.0/
      DO 10 I=1,30
 10   DELT(I)=0.01
C END OF PROBLEM-DEPENDANT DATA
```

NUMERICAL EVALUATION OF DUHAMEL INTEGRAL

I	T(I)	F(I)	X(I)
2	0.99999998E-02	0.60000000E+02	0.14990796E-02
3	0.20000000E-01	0.60000000E+02	0.59849964E-02
4	0.29999999E-01	0.60000000E+02	0.13424230E-01
5	0.39999999E-01	0.60000000E+02	0.23760952E-01
6	0.50000001E-01	0.10000000E+03	0.37250735E-01
7	0.60000002E-01	0.10000000E+03	0.55125091E-01
8	0.70000000E-01	0.10000000E+03	0.77583194E-01
9	0.79999998E-01	0.10000000E+03	0.10445669E+00
10	0.89999996E-01	0.10000000E+03	0.13554408E+00
11	0.99999994E-01	0.30000000E+02	0.17002963E+00
12	0.10999999E+00	0.30000000E+02	0.20532255E+00
13	0.11999999E+00	0.30000000E+02	0.24057558E+00
14	0.13000000E+00	0.30000000E+02	0.27552453E+00
15	0.14000000E+00	0.30000000E+02	0.30990744E+00
16	0.15000001E+00	0.30000000E+02	0.34346649E+00
17	0.16000001E+00	0.00000000E+00	0.37570029E+00
18	0.17000002E+00	0.00000000E+00	0.40536803E+00
19	0.18000002E+00	0.00000000E+00	0.43199742E+00
20	0.19000003E+00	0.00000000E+00	0.45538884E+00
21	0.20000003E+00	0.00000000E+00	0.47536695E+00
22	0.21000004E+00	0.00000000E+00	0.49178207E+00
23	0.22000004E+00	0.00000000E+00	0.50451112E+00
24	0.23000005E+00	0.00000000E+00	0.51345861E+00
25	0.24000005E+00	0.00000000E+00	0.51855767E+00
26	0.25000006E+00	0.00000000E+00	0.51976985E+00
27	0.26000005E+00	0.00000000E+00	0.51708633E+00
28	0.27000004E+00	0.00000000E+00	0.51052707E+00
29	0.28000003E+00	0.00000000E+00	0.50914120E+00
30	0.29000002E+00	0.00000000E+00	0.48600671E+00

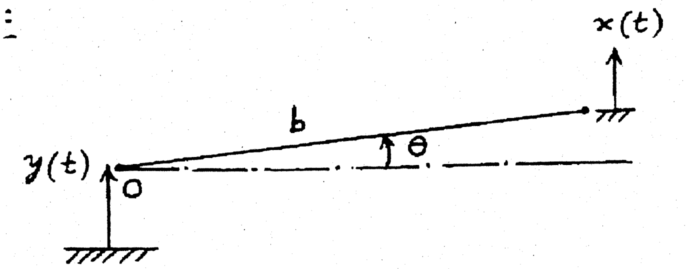
(4.58) (i) Find natural frequency:

$$J_o = m b^2 = 9 m a^2$$

$$\frac{1}{2} k_t \theta^2 = \frac{1}{2} k (a\theta)^2 = \frac{1}{2} k a^2 \theta^2 \Rightarrow k_t = k a^2$$

$$\omega_n = \sqrt{\frac{k_t}{J_o}} = \sqrt{\frac{k a^2}{9 m a^2}} = \frac{1}{3}\sqrt{\frac{k}{m}} = 2\pi(10) = 20\pi \ \frac{\text{rad}}{\text{sec}} \qquad (E_1)$$

(ii) Find equation of motion:

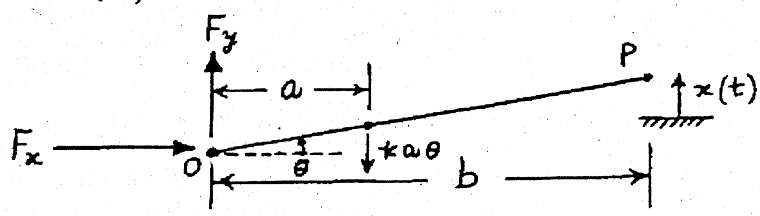


when the base and hence the pivot O is displaced by $y(t)$, the displacement of mass m, $x(t)$, is given by

$$x(t) = y(t) + b\,\theta(t) \qquad (E_2)$$

Relative displacement of mass is

$$z(t) = x(t) - y(t) = b\,\theta(t) \qquad (E_3)$$



Equation of motion:

Σ Forces along y (vertical) direction $= 0 \Rightarrow F_y - k a \theta = m\ddot{x} \qquad (E_4)$

Σ Moments about P $= 0 \Rightarrow F_y \cdot b - k a \theta (b - a) = 0 \qquad (E_5)$

Solve (E_4) for F_y and substitute the result in (E_5):

$$(k a \theta + m\ddot{x})\, b - k a \theta (b - a) = 0$$

i.e., $$m b \ddot{x} + \theta k a^2 = 0 \qquad (E_6)$$

But $x = y + z$ and $\theta(t) = \frac{1}{b} z(t)$

Hence (E_6) becomes

$$m b \ddot{z} + \frac{k a^2}{b} z = - m b \ddot{y}$$

or $$m \ddot{z} + \frac{k a^2}{b^2} z = - m \ddot{y} \qquad (E_7)$$

Eq. (E_7) can be compared with a standard forced vibration equation for an undamped system:

$$\underset{\sim}{m}\ \ddot{x} + \underset{\sim}{k}\ x = \underset{\sim}{F}(t) \qquad (E_8)$$

Comparison of (E_7) and (E_8) shows that

$$x = z, \quad \underset{\sim}{m} = m, \quad \underset{\sim}{k} = \frac{k a^2}{b^2} \quad \text{and} \quad \underset{\sim}{F} = -m\ \ddot{y} \qquad (E_9)$$

(iii) Find solution:

Solution of Eq. (E_8) under a rectangular pulse is given in problem 4.19. Hence the solution of problem 4.19 can be used to find the relative displacement.

(iv) Design:

Eq. (E_1) gives $\quad \sqrt{k} = 60\pi\ \sqrt{m} \quad$ or $\quad k = 3600\ \pi^2\ m \qquad (E_{10})$

Following procedure can be used to solve the problem:

a. Assume a value of a.
b. Assume a small value of m.
c. Find k from Eq. (E_{10}).
d. Evaluate the solution numerically as outlined in part (iii).
e. If the maximum relative displacement is larger than or equal to 0.02 m, the design is complete.
f. Otherwise, increase the value of m and go to step c.
g. If necessary change the value of a in step a.

4.59 Let k and m denote the equivalent stiffness and mass of the cutting head.

Equation of motion is: $\quad m\ddot{x} + k x = F(t) \qquad (E_1)$

where $F(t)$ is given by Figs. 4.34 (a) and (b).

Solution of Eq. (E_1) under the force given by Fig. 4.56 (a) can be obtained as in problem 4.15. Solution of (E_1) under the force of Fig. 4.56 (b) can be determined using a procedure similar to that of problem 4.24.

The values of k and m can be determined as follows;

a. Assume a small value for m.
b. Assume a small value for k.
c. Evaluate the response under $F(t)$ given by Fig. 4.56(a).

d. Evaluate the response under $F(t)$ given by Fig. 4.56 (b).

e. If the responses in steps c and d are approximately equal to 0.1 mm and 0.05 mm, the current values of m and k are the desired values.

f. Otherwise, increment the value of m and/or k and go to step c.

(4.60) Model the system as a single d.o.f. torsional system with:

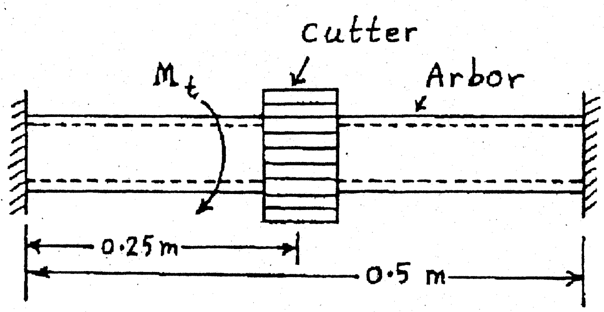
$J_0 = 0.1$ N–m^2 ; $c_t = 0$ (no damping assumed) ;

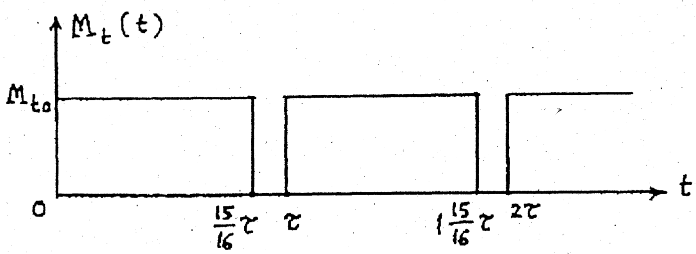
$$k_t = \frac{G\,J}{\ell} = \frac{1}{0.25}\,(80\,(10^9))\left(\frac{\pi}{32}\,(d_o^4 - d_i^4)\right)$$

where d_o = outer diameter of shaft, and d_i = inner diameter.

$$k_t = 62.832\,(10^9)\,(d_o^4 - d_i^4) \tag{1}$$

Torque acting on the arbor due to breakage of one tooth can be modeled as shown in figure.





$M_{to} = 500$ N-m, $\tau = \dfrac{2\,\pi}{\omega} = \dfrac{2\,\pi}{N\,(2\,\pi)/60} = 60/N = 60/1000 = 0.06$ sec where N = speed of cutter = 1000 rpm. Express $M_t(t)$ in Fourier series:

$$M_t(t) = \frac{a_0}{2} + \sum_{n=1}^{\infty}\left(a_n \cos n\,\omega\,t + b_n \sin n\,\omega\,t\right) \tag{2}$$

(see solution of Problem 4.6 for procedure). where $\omega = \dfrac{2\,\pi\,N}{60} = 104.72$ rad/sec.

Equation of motion:

$$J_0\,\ddot{\theta} + k_t\,\theta = M_t(t) \tag{3}$$

where $M_t(t)$ is given by Eq. (2). Find solution of Eq. (3) and determine the maximum value of θ, θ_{max}. This value must be less than 1°.

Note:
The solution requires an iterative procedure. Assume values for d_i and d_o. Compute k_t, find solution of $\theta(t)$, and the value of θ_{max}. If θ_{max} exceeds 1°, choose a different set of values for d_i and d_o and continue the process until θ_{max} comes out to be less than 1°.