CHAPTER
10

TIME-VARYING FIELDS AND MAXWELL'S EQUATIONS

The basic relationships of the electrostatic and the steady magnetic field were obtained in the previous nine chapters, and we are now ready to discuss time-varying fields. The discussion will be short, for vector analysis and vector calculus should now be more familiar tools; some of the relationships are unchanged, and most of the relationships are changed only slightly.

Two new concepts will be introduced: the electric field produced by a changing magnetic field and the magnetic field produced by a changing electric field. The first of these concepts resulted from experimental research by Michael Faraday, and the second from the theoretical efforts of James Clerk Maxwell.

Maxwell actually was inspired by Faraday's experimental work and by the mental picture provided through the "lines of force" that Faraday introduced in developing his theory of electricity and magnetism. He was 40 years younger than Faraday, but they knew each other during the 5 years Maxwell spent in London as a young professor, a few years after Faraday had retired. Maxwell's theory was developed subsequent to his holding this university position, while he was working alone at his home in Scotland. It occupied him for 5 years between the ages of 35 and 40.

The four basic equations of electromagnetic theory presented in this chapter bear his name.

10.1 FARADAY'S LAW

After Oersted[1] demonstrated in 1820 that an electric current affected a compass needle, Faraday professed his belief that if a current could produce a magnetic field, then a magnetic field should be able to produce a current. The concept of the "field" was not available at that time, and Faraday's goal was to show that a current could be produced by "magnetism."

He worked on this problem intermittently over a period of ten years, until he was finally successful in 1831.[2] He wound two separate windings on an iron toroid and placed a galvanometer in one circuit and a battery in the other. Upon closing the battery circuit, he noted a momentary deflection of the galvanometer; a similar deflection in the opposite direction occurred when the battery was disconnected. This, of course, was the first experiment he made involving a *changing* magnetic field, and he followed it with a demonstration that either a *moving* magnetic field or a moving coil could also produce a galvanometer deflection.

In terms of fields, we now say that a time-varying magnetic field produces an *electromotive force* (emf) which may establish a current in a suitable closed circuit. An electromotive force is merely a voltage that arises from conductors moving in a magnetic field or from changing magnetic fields, and we shall define it below. Faraday's law is customarily stated as

$$\boxed{\text{emf} = -\frac{d\Phi}{dt} \quad \text{V}} \tag{1}$$

Equation (1) implies a closed path, although not necessarily a closed conducting path; the closed path, for example, might include a capacitor, or it might be a purely imaginary line in space. The magnetic flux is that flux which passes through any and every surface whose perimeter is the closed path, and $d\Phi/dt$ is the time rate of change of this flux.

A nonzero value of $d\Phi/dt$ may result from any of the following situations:

1. A time-changing flux linking a stationary closed path
2. Relative motion between a steady flux and a closed path
3. A combination of the two

The minus sign is an indication that the emf is in such a direction as to produce a current whose flux, if added to the original flux, would reduce the magnitude of the emf. This statement that the induced voltage acts to produce an opposing flux is known as *Lenz's law*.[3]

[1] Hans Christian Oersted was Professor of Physics at the University of Copenhagen in Denmark.

[2] Joseph Henry produced similar results at Albany Academy in New York at about the same time.

[3] Henri Frederic Emile Lenz was born in Germany but worked in Russia. He published his law in 1834.

If the closed path is that taken by an N-turn filamentary conductor, it is often sufficiently accurate to consider the turns as coincident and let

$$\boxed{\text{emf} = -N\frac{d\Phi}{dt}} \tag{2}$$

where Φ is now interpreted as the flux passing through any one of N coincident paths.

We need to define emf as used in (1) or (2). The emf is obviously a scalar, and (perhaps not so obviously) a dimensional check shows that it is measured in volts. We define the emf as

$$\boxed{\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L}} \tag{3}$$

and note that it is the voltage about a specific *closed path*. If any part of the path is changed, generally the emf changes. The departure from static results is clearly shown by (3), for an electric field intensity resulting from a static charge distribution must lead to zero potential difference about a closed path. In electrostatics, the line integral leads to a potential difference; with time-varying fields, the result is an emf or a voltage.

Replacing Φ in (1) by the surface integral of $\mathbf{B}$, we have

$$\boxed{\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\frac{d}{dt}\int_S \mathbf{B} \cdot d\mathbf{S}} \tag{4}$$

where the fingers of our right hand indicate the direction of the closed path, and our thumb indicates the direction of $d\mathbf{S}$. A flux density $\mathbf{B}$ in the direction of $d\mathbf{S}$ and increasing with time thus produces an average value of $\mathbf{E}$ which is *opposite* to the positive direction about the closed path. The right-handed relationship between the surface integral and the closed line integral in (4) should always be kept in mind during flux integrations and emf determinations.

Let us divide our investigation into two parts by first finding the contribution to the total emf made by a changing field within a stationary path (transformer emf), and then we will consider a moving path within a constant (motional, or generator, emf).

We first consider a stationary path. The magnetic flux is the only time-varying quantity on the right side of (4), and a partial derivative may be taken under the integral sign,

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} \tag{5}$$

Before we apply this simple result to an example, let us obtain the point form of this integral equation. Applying Stokes' theorem to the closed line integral, we have

$$\int_S (\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

where the surface integrals may be taken over identical surfaces. The surfaces are perfectly general and may be chosen as differentials,

$$(\nabla \times \mathbf{E}) \cdot d\mathbf{S} = -\frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}$$

and

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{6}$$

This is one of Maxwell's four equations as written in differential, or point, form, the form in which they are most generally used. Equation (5) is the integral form of this equation and is equivalent to Faraday's law as applied to a fixed path. If **B** is not a function of time, (5) and (6) evidently reduce to the electrostatic equations,

$$\oint \mathbf{E} \cdot d\mathbf{L} = 0 \quad \text{(electrostatics)}$$

and

$$\nabla \times \mathbf{E} = 0 \quad \text{(electrostatics)}$$

As an example of the interpretation of (5) and (6), let us assume a simple magnetic field which increases exponentially with time within the cylindrical region $\rho < b$,

$$\mathbf{B} = B_0 e^{kt} \mathbf{a}_z \tag{7}$$

where $B_0 =$ constant. Choosing the circular path $\rho = a$, $a < b$, in the $z = 0$ plane, along which E_ϕ must be constant by symmetry, we then have from (5)

$$\text{emf} = 2\pi a E_\phi = -kB_0 e^{kt} \pi a^2$$

The emf around this closed path is $-kB_0 e^{kt} \pi a^2$. It is proportional to a^2, because the magnetic flux density is uniform and the flux passing through the surface at any instant is proportional to the area.

If we now replace a by ρ, $\rho < b$, the electric field intensity at any point is

$$\mathbf{E} = -\tfrac{1}{2} k B_0 e^{kt} \rho \mathbf{a}_\phi \tag{8}$$

Let us now attempt to obtain the same answer from (6), which becomes

$$(\nabla \times \mathbf{E})_z = -kB_0 e^{kt} = \frac{1}{\rho}\frac{\partial(\rho E_\phi)}{\partial \rho}$$

Multiplying by ρ and integrating from 0 to ρ (treating t as a constant, since the derivative is a partial derivative),

$$-\tfrac{1}{2}kB_0 e^{kt}\rho^2 = \rho E_\phi$$

or

$$\mathbf{E} = -\tfrac{1}{2}kB_0 e^{kt}\rho\mathbf{a}_\phi$$

once again.

If B_0 is considered positive, a filamentary conductor of resistance R would have a current flowing in the negative $\mathbf{a}_\phi$ direction, and this current would establish a flux within the circular loop in the negative $\mathbf{a}_z$ direction. Since E_ϕ increases exponentially with time, the current and flux do also, and thus tend to reduce the time rate of increase of the applied flux and the resultant emf in accordance with Lenz's law.

Before leaving this example, it is well to point out that the given field **B** does not satisfy all of Maxwell's equations. Such fields are often assumed (*always* in ac-circuit problems) and cause no difficulty when they are interpreted properly. They occasionally cause surprise, however. This particular field is discussed further in Prob. 19 at the end of the chapter.

Now let us consider the case of a time-constant flux and a moving closed path. Before we derive any special results from Faraday's law (1), let us use the basic law to analyze the specific problem outlined in Fig. 10.1. The closed circuit consists of two parallel conductors which are connected at one end by a high-resistance voltmeter of negligible dimensions and at the other end by a sliding bar moving at a velocity **v**. The magnetic flux density **B** is constant (in space and time) and is normal to the plane containing the closed path.

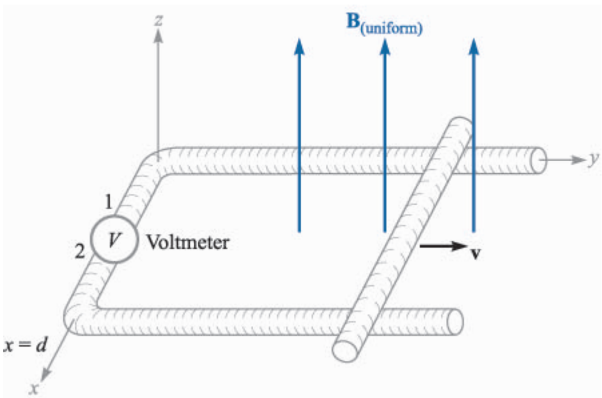

FIGURE 10.1
An example illustrating the application of Faraday's law to the case of a constant magnetic flux density **B** and a moving path. The shorting bar moves to the right with a velocity **v**, and the circuit is completed through the two rails and an extremely small high-resistance voltmeter. The voltmeter reading is $V_{12} = -Bvd$.

Let the position of the shorting bar be given by y; the flux passing through the surface within the closed path at any time t is then

$$\Phi = Byd$$

From (1), we obtain

$$\text{emf} = -\frac{d\Phi}{dt} = -B\frac{dy}{dt}d = -Bvd \tag{9}$$

The emf is defined as $\oint \mathbf{E} \cdot d\mathbf{L}$ and we have a conducting path; so we may actually determine $\mathbf{E}$ at every point along the closed path. We found in electrostatics that the tangential component of $\mathbf{E}$ is zero at the surface of a conductor, and we shall show in Sec. 10.4 that the tangential component is zero at the surface of a *perfect* conductor ($\sigma = \infty$) for all time-varying conditions. This is equivalent to saying that a perfect conductor is a "short circuit." The entire closed path in Figure 10.1 may be considered as a perfect conductor, with the exception of the voltmeter. The actual computation of $\oint \mathbf{E} \cdot d\mathbf{L}$ then must involve no contribution along the entire moving bar, both rails, and the voltmeter leads. Since we are integrating in a counterclockwise direction (keeping the interior of the positive side of the surface on our left as usual), the contribution $E\,\Delta L$ across the voltmeter must be $-Bvd$, showing that the electric field intensity in the instrument is directed from terminal 2 to terminal 1. For an up-scale reading, the positive terminal of the voltmeter should therefore be terminal 2.

The direction of the resultant small current flow may be confirmed by noting that the enclosed flux is reduced by a clockwise current in accordance with Lenz's law. The voltmeter terminal 2 is again seen to be the positive terminal.

Let us now consider this example using the concept of *motional emf*. The force on a charge Q moving at a velocity $\mathbf{v}$ in a magnetic field $\mathbf{B}$ is

$$\boxed{\mathbf{F} = Q\mathbf{v} \times \mathbf{B}}$$

or

$$\frac{\mathbf{F}}{Q} = \mathbf{v} \times \mathbf{B} \tag{10}$$

The sliding conducting bar is composed of positive and negative charges, and each experiences this force. The force per unit charge, as given by (10), is called the *motional* electric field intensity $\mathbf{E}_m$,

$$\boxed{\mathbf{E}_m = \mathbf{v} \times \mathbf{B}} \tag{11}$$

If the moving conductor were lifted off the rails, this electric field intensity would force electrons to one end of the bar (the far end) until the *static field* due to these

charges just balanced the field induced by the motion of the bar. The resultant tangential electric field intensity would then be zero along the length of the bar.

The motional emf produced by the moving conductor is then

$$\text{emf} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{12}$$

where the last integral may have a nonzero value only along that portion of the path which is in motion, or along which $\mathbf{v}$ has some nonzero value. Evaluating the right side of (12), we obtain

$$\oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} = \int_d^0 vB\,dx = -Bvd$$

as before. This is the total emf, since $\mathbf{B}$ is not a function of time.

In the case of a conductor moving in a uniform constant magnetic field, we may therefore ascribe a motional electric field intensity $\mathbf{E}_m = \mathbf{v} \times \mathbf{B}$ to every portion of the moving conductor and evaluate the resultant emf by

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = \oint \mathbf{E}_m \cdot d\mathbf{L} = \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{13}$$

If the magnetic flux density is also changing with time, then we must include both contributions, the transformer emf (5) and the motional emf (12),

$$\text{emf} = \oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S} + \oint (\mathbf{v} \times \mathbf{B}) \cdot d\mathbf{L} \tag{14}$$

This expression is equivalent to the simple statement

$$\text{emf} = -\frac{d\Phi}{dt} \tag{1}$$

and either can be used to determine these induced voltages.

Although (1) appears simple, there are a few contrived examples in which its proper application is quite difficult. These usually involve sliding contacts or switches; they always involve the substitution of one part of a circuit by a new part.[4] As an example, consider the simple circuit of Fig. 10.2, containing several perfectly conducting wires, an ideal voltmeter, a uniform constant field $\mathbf{B}$, and a switch. When the switch is opened, there is obviously more flux enclosed in the voltmeter circuit; however, it continues to read zero. The change in flux has not been produced by either a time-changing $\mathbf{B}$ [first term of (14)] or a conductor moving through a magnetic field [second part of (14)]. Instead, a new circuit has been substituted for the old. Thus it is necessary to use care in evaluating the change in flux linkages.

The separation of the emf into the two parts indicated by (14), one due to the time rate of change of $\mathbf{B}$ and the other to the motion of the circuit, is some-

[4] See Bewley, in Suggested References at the end of the chapter, particularly pp. 12–19.

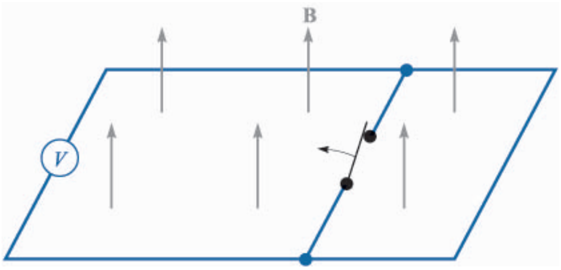

FIGURE 10.2
An apparent increase in flux linkages does not lead to an induced voltage when one part of a circuit is simply substituted for another by opening the switch. No indication will be observed on the voltmeter.

what arbitrary in that it depends on the relative velocity of the *observer* and the system. A field that is changing with both time and space may look constant to an observer moving with the field. This line of reasoning is developed more fully in applying the special theory of relativity to electromagnetic theory.[5]

✔ **D10.1.** Within a certain region, $\epsilon = 10^{-11}$ F/m and $\mu = 10^{-5}$ H/m. If $B_x = 2 \times 10^{-4} \cos 10^5 t \sin 10^{-3} y$ T: (*a*) use $\nabla \times \mathbf{H} = \epsilon \dfrac{\partial \mathbf{E}}{\partial t}$ to find **E**; (*b*) find the total magnetic flux passing through the surface $x = 0$, $0 < y < 40$ m, $0 < z < 2$ m, at $t = 1\ \mu$s; (*c*) find the value of the closed line integral of **E** around the perimeter of the given surface.

Ans. $-20\,000 \sin 10^5 t \cos 10^{-3} y \mathbf{a}_z$ V/m; 31.4 mWb; -315 V

✔ **D10.2.** With reference to the sliding bar shown in Figure 10.1, let $d = 7$ cm, $\mathbf{B} = 0.3\mathbf{a}_z$ T, and $\mathbf{v} = 0.1\mathbf{a}_y e^{20y}$ m/s. Let $y = 0$ at $t = 0$. Find: (*a*) $v(t = 0)$; (*b*) $y(t = 0.1)$; (*c*) $v(t = 0.1)$; (*d*) V_{12} at $t = 0.1$.

Ans. 0.1 m/s; 1.116 cm; 0.1250 m/s; -0.002625 V

10.2 DISPLACEMENT CURRENT

Faraday's experimental law has been used to obtain one of Maxwell's equations in differential form,

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{15}$$

which shows us that a time-changing magnetic field produces an electric field. Remembering the definition of curl, we see that this electric field has the special property of circulation; its line integral about a general closed path is not zero. Now let us turn our attention to the time-changing electric field.

We should first look at the point form of Ampère's circuital law as it applies to steady magnetic fields,

[5] This is discussed in several of the references listed in the Suggested References at the end of the chapter. See Panofsky and Phillips, pp. 142–151; Owen, pp. 231–245; and Harman in several places.

$$\nabla \times \mathbf{H} = \mathbf{J} \tag{16}$$

and show its inadequacy for time-varying conditions by taking the divergence of each side,

$$\nabla \cdot \nabla \times \mathbf{H} \equiv 0 = \nabla \cdot \mathbf{J}$$

The divergence of the curl is identically zero, so $\nabla \cdot \mathbf{J}$ is also zero. However, the equation of continuity,

$$\nabla \cdot \mathbf{J} = -\frac{\partial \rho_v}{\partial t}$$

then shows us that (16) can be true only if $\partial \rho_v / \partial t = 0$. This is an unrealistic limitation, and (16) must be amended before we can accept it for time-varying fields. Suppose we add an unknown term $\mathbf{G}$ to (16),

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{G}$$

Again taking the divergence, we have

$$0 = \nabla \cdot \mathbf{J} + \nabla \cdot \mathbf{G}$$

Thus

$$\nabla \cdot \mathbf{G} = \frac{\partial \rho_v}{\partial t}$$

Replacing ρ_v by $\nabla \cdot \mathbf{D}$,

$$\nabla \cdot \mathbf{G} = \frac{\partial}{\partial t}(\nabla \cdot \mathbf{D}) = \nabla \cdot \frac{\partial \mathbf{D}}{\partial t}$$

from which we obtain the simplest solution for $\mathbf{G}$,

$$\mathbf{G} = \frac{\partial \mathbf{D}}{\partial t}$$

Ampère's circuital law in point form therefore becomes

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{17}$$

Equation (17) has not been derived. It is merely a form we have obtained which does not disagree with the continuity equation. It is also consistent with all our other results, and we accept it as we did each experimental law and the equations derived from it. We are building a theory, and we have every right to our equations *until they are proved wrong*. This has not yet been done.

We now have a second one of Maxwell's equations and shall investigate its significance. The additional term $\partial \mathbf{D}/\partial t$ has the dimensions of current density, amperes per square meter. Since it results from a time-varying electric flux density (or displacement density), Maxwell termed it a *displacement current density*. We sometimes denote it by $\mathbf{J}_d$:

$$\nabla \times \mathbf{H} = \mathbf{J} + \mathbf{J}_d$$

$$\mathbf{J}_d = \frac{\partial \mathbf{D}}{\partial t}$$

This is the third type of current density we have met. Conduction current density,

$$\mathbf{J} = \sigma \mathbf{E}$$

is the motion of charge (usually electrons) in a region of zero net charge density, and convection current density,

$$\mathbf{J} = \rho_v \mathbf{v}$$

is the motion of volume charge density. Both are represented by **J** in (17). Bound current density is, of course, included in **H**. In a nonconducting medium in which no volume charge density is present, $\mathbf{J} = 0$, and then

$$\nabla \times \mathbf{H} = \frac{\partial \mathbf{D}}{\partial t} \quad (\text{if } \mathbf{J} = 0) \tag{18}$$

Notice the symmetry between (18) and (15):

$$\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t} \tag{15}$$

Again the analogy between the intensity vectors **E** and **H** and the flux density vectors **D** and **B** is apparent. Too much faith cannot be placed in this analogy, however, for it fails when we investigate forces on particles. The force on a charge is related to **E** and to **B**, and some good arguments may be presented showing an analogy between **E** and **B** and between **D** and **H**. We shall omit them, however, and merely say that the concept of displacement current was probably suggested to Maxwell by the symmetry first mentioned above.[6]

The total displacement current crossing any given surface is expressed by the surface integral,

$$I_d = \int_S \mathbf{J}_d \cdot d\mathbf{S} = \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

and we may obtain the time-varying version of Ampère's circuital law by integrating (17) over the surface S,

$$\int_S (\nabla \times \mathbf{H}) \cdot d\mathbf{S} = \int_S \mathbf{J} \cdot d\mathbf{S} + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}$$

[6] The analogy that relates **B** to **D** and **H** to **E** is strongly advocated by Fano, Chu, and Adler (see Suggested References for Chap. 5) on pp. 159–160 and 179; the case for comparing **B** to **E** and **D** to **H** is presented in Halliday and Resnick (see Suggested References for this chapter) on pp. 665–668 and 832–836.

and applying Stokes' theorem,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I + I_d = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}} \tag{19}$$

What is the nature of displacement current density? Let us study the simple circuit of Fig. 10.3, containing a filamentary loop and a parallel-plate capacitor. Within the loop a magnetic field varying sinusoidally with time is applied to produce an emf about the closed path (the filament plus the dashed portion between the capacitor plates) which we shall take as

$$\text{emf} = V_0 \cos \omega t$$

Using elementary circuit theory and assuming the loop has negligible resistance and inductance, we may obtain the current in the loop as

$$I = -\omega C V_0 \sin \omega t$$
$$= -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$$

where the quantities ϵ, S, and d pertain to the capacitor. Let us apply Ampère's circuital law about the smaller closed circular path k and neglect displacement current for the moment:

$$\oint_k \mathbf{H} \cdot d\mathbf{L} = I_k$$

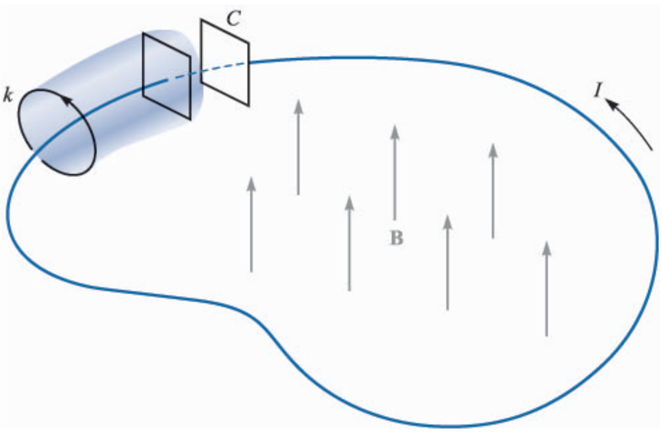

FIGURE 10.3
A filamentary conductor forms a loop connecting the two plates of a parallel-plate capacitor. A time-varying magnetic field inside the closed path produces an emf of $V_0 \cos \omega t$ around the closed path. The conduction current I is equal to the displacement current between the capacitor plates.

The path and the value of **H** along the path are both definite quantities (although difficult to determine), and $\oint_k \mathbf{H} \cdot d\mathbf{L}$ is a definite quantity. The current I_k is that current through every surface whose perimeter is the path k. If we choose a simple surface punctured by the filament, such as the plane circular surface defined by the circular path k, the current is evidently the conduction current. Suppose now we consider the closed path k as the mouth of a paper bag whose bottom passes between the capacitor plates. The bag is not pierced by the filament, and the conductor current is zero. Now we need to consider displacement current, for within the capacitor

$$D = \epsilon E = \epsilon\left(\frac{V_0}{d}\cos \omega t\right)$$

and therefore

$$I_d = \frac{\partial D}{\partial t} S = -\omega \frac{\epsilon S}{d} V_0 \sin \omega t$$

This is the same value as that of the conduction current in the filamentary loop. Therefore the applicaton of Ampère's circuital law including displacement current to the path k leads to a definite value for the line integral of **H**. This value must be equal to the total current crossing the chosen surface. For some surfaces the current is almost entirely conduction current, but for those surfaces passing between the capacitor plates, the conduction current is zero, and it is the displacement current which is now equal to the closed line integral of **H**.

Physically, we should note that a capacitor stores charge and that the electric field between the capacitor plates is much greater than the small leakage fields outside. We therefore introduce little error when we neglect displacement current on all those surfaces which do not pass between the plates.

Displacement current is associated with time-varying electric fields and therefore exists in all imperfect conductors carrying a time-varying conduction current. The last part of the drill problem below indicates the reason why this additional current was never discovered experimentally. This comparison is illustrated further in Sec. 11.3.

✔ **D10.3.** Find the amplitude of the displacement current density: (*a*) adjacent to an automobile antenna where the magnetic field intensity of an FM signal is $H_x = 0.15\cos[3.12(3 \times 10^8 t - y)]$ A/m; (*b*) in the air space at a point within a large power distribution transformer where $\mathbf{B} = 0.8\cos[1.257 \times 10^{-6}(3 \times 10^8 t - x)]\mathbf{a}_y$ T; (*c*) within a large oil-filled power capacitor where $\epsilon_R = 5$ and $\mathbf{E} = 0.9\cos[1.257 \times 10^{-6}(3 \times 10^8 t - z\sqrt{5})]\mathbf{a}_x$ MV/m; (*d*) in a metallic conductor at 60 Hz, if $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 5.8 \times 10^7$ S/m, and $\mathbf{J} = \sin(377t - 117.1z)\mathbf{a}_x$ MA/m^2.

Ans. 0.318 A/m^2; 0.800 A/m^2; 0.01502 A/m^2; 57.6 pA/m^2

10.3 MAXWELL'S EQUATIONS IN POINT FORM

We have already obtained two of Maxwell's equations for time-varying fields,

$$\boxed{\nabla \times \mathbf{E} = -\frac{\partial \mathbf{B}}{\partial t}} \tag{20}$$

and

$$\boxed{\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}} \tag{21}$$

The remaining two equations are unchanged from their non-time-varying form:

$$\boxed{\nabla \cdot \mathbf{D} = \rho_v} \tag{22}$$

$$\boxed{\nabla \cdot \mathbf{B} = 0} \tag{23}$$

Equation (22) essentially states that charge density is a source (or sink) of electric flux lines. Note that we can no longer say that *all* electric flux begins and terminates on charge, because the point form of Faraday's law (20) shows that **E**, and hence **D**, may have circulation if a changing magnetic field is present. Thus the lines of electric flux may form closed loops. However, the converse is still true, and every coulomb of charge must have one coulomb of electric flux diverging from it.

Equation (23) again acknowledges the fact that "magnetic charges," or poles, are not known to exist. Magnetic flux is always found in closed loops and never diverges from a point source.

These for equations form the basis of all electromagnetic theory. They are partial differential equations and relate the electric and magnetic fields to each other and to their sources, charge and current density. The auxiliary equations relating **D** and **E**.

$$\boxed{\mathbf{D} = \epsilon \mathbf{E}} \tag{24}$$

relating **B** and **H**,

$$\boxed{\mathbf{B} = \mu \mathbf{H}} \tag{25}$$

defining conduction current density,

$$\boxed{\mathbf{J} = \sigma \mathbf{E}} \tag{26}$$

and defining convection current density in terms of the volume charge density ρ_v,

$$\boxed{\mathbf{J} = \rho_v \mathbf{v}} \tag{27}$$

are also required to define and relate the quantities appearing in Maxwell's equations.

The potentials V and $\mathbf{A}$ have not been included above because they are not strictly necessary, although they are extremely useful. They will be discussed at the end of this chapter.

If we do not have "nice" materials to work with, then we should replace (24) and (25) by the relationships involving the polarization and magnetization fields,

$$\boxed{\mathbf{D} = \epsilon_0 \mathbf{E} + \mathbf{P}} \tag{28}$$

$$\boxed{\mathbf{B} = \mu_0 (\mathbf{H} + \mathbf{M})} \tag{29}$$

For linear materials we may relate **P** to **E**

$$\mathbf{P} = \chi_e \epsilon_0 \mathbf{E} \tag{30}$$

and **M** to **H**

$$\mathbf{M} = \chi_m \mathbf{H} \tag{31}$$

Finally, because of its fundamental importance we should include the Lorentz force equation, written in point form as the force per unit volume,

$$\boxed{\mathbf{f} = \rho_v (\mathbf{E} + \mathbf{v} \times \mathbf{B})} \tag{32}$$

The following chapters are devoted to the application of Maxwell's equations to several simple problems.

✔ **D10.4.** Let $\mu = 10^{-5}$ H/m, $\epsilon = 4 \times 10^{-9}$ F/m, $\sigma = 0$, and $\rho_v = 0$. Find k (including units) so that each of the following pairs of fields satisfies Maxwell's equations: (*a*) $\mathbf{D} = 6\mathbf{a}_x - 2y\mathbf{a}_y + 2z\mathbf{a}_z$ nC/m^2, $\mathbf{H} = kx\mathbf{a}_x + 10y\mathbf{a}_y - 25z\mathbf{a}_z$ A/m; (*b*) $\mathbf{E} = (20y - kt)\mathbf{a}_x$ V/m, $\mathbf{H} = (y + 2 \times 10^6 t)\mathbf{a}_z$ A/m.

Ans. 15 A/m^2; -2.5×10^8 V/(m·s)

10.4 MAXWELL'S EQUATIONS IN INTEGRAL FORM

The integral forms of Maxwell's equations are usually easier to recognize in terms of the experimental laws from which they have been obtained by a generalization process. Experiments must treat physical macroscopic quantities, and their results therefore are expressed in terms of integral relationships. A differential equation always represents a theory. Let us now collect the integral forms of Maxwell's equations of the previous section.

Integrating (20) over a surface and applying Stokes' theorem, we obtain Faraday's law,

$$\boxed{\oint \mathbf{E} \cdot d\mathbf{L} = -\int_S \frac{\partial \mathbf{B}}{\partial t} \cdot d\mathbf{S}} \tag{33}$$

and the same process applied to (21) yields Ampère's circuital law,

$$\boxed{\oint \mathbf{H} \cdot d\mathbf{L} = I + \int_S \frac{\partial \mathbf{D}}{\partial t} \cdot d\mathbf{S}} \tag{34}$$

Gauss's laws for the electric and magnetic fields are obtained by integrating (22) and (23) throughout a volume and using the divergence theorem:

$$\boxed{\oint_S \mathbf{D} \cdot d\mathbf{S} = \int_{\text{vol}} \rho_v dv} \tag{35}$$

$$\boxed{\oint_S \mathbf{B} \cdot d\mathbf{S} = 0} \tag{36}$$

These four integral equations enable us to find the boundary conditions on **B**, **D**, **H**, and **E** which are necessary to evaluate the constants obtained in solving Maxwell's equations in partial differential form. These boundary conditions are in general unchanged from their forms for static or steady fields, and the same methods may be used to obtain them. Between any two real physical media (where **K** must be zero on the boundary surface), (33) enables us to relate the tangential **E**-field components,

$$E_{t1} = E_{t2} \tag{37}$$

and from (34),

$$H_{t1} = H_{t2} \tag{38}$$

The surface integrals produce the boundary conditions on the normal components,

$$D_{N1} - D_{N2} = \rho_S \tag{39}$$

and

$$B_{N1} = B_{N2} \tag{40}$$

It is often desirable to idealize a physical problem by assuming a perfect conductor for which σ is infinite but $\mathbf{J}$ is finite. From Ohm's law, then, in a perfect conductor,

$$\mathbf{E} = 0$$

and it follows from the point form of Faraday's law that

$$\mathbf{H} = 0$$

for time-varying fields. The point form of Ampère's circuital law then shows that the finite value of $\mathbf{J}$ is

$$\mathbf{J} = 0$$

and current must be carried on the conductor surface as a surface current $\mathbf{K}$. Thus, if region 2 is a perfect conductor, (37) to (40) become, respectively,

$$E_{t1} = 0 \tag{41}$$

$$H_{t1} = K \quad (\mathbf{H}_{t1} = \mathbf{K} \times \mathbf{a}_N) \tag{42}$$

$$D_{N1} = \rho_s \tag{43}$$

$$B_{N1} = 0 \tag{44}$$

where $\mathbf{a}_N$ is an outward normal at the conductor surface.

Note that surface charge density is considered a physical possibility for either dielectrics, perfect conductors, or imperfect conductors, but that surface *current* density is assumed only in conjunction with perfect conductors.

The boundary conditions stated above are a very necessary part of Maxwell's equations. All real physical problems have boundaries and require the solution of Maxwell's equations in two or more regions and the matching of these solutions at the boundaries. In the case of perfect conductors, the solution of the equations within the conductor is trivial (all time-varying fields are zero), but the application of the boundary conditions (41) to (44) may be very difficult.

Certain fundamental properties of wave propagation are evident when Maxwell's equations are solved for an *unbounded* region. This problem is treated in the following chapter. It represents the simplest application of Maxwell's equations, becuase it is the only problem which does not require the application of any boundary conditions.

✔ **D10.5.** The unit vector $0.64\mathbf{a}_x + 0.6\mathbf{a}_y - 0.48\mathbf{a}_z$ is directed from region 2 ($\epsilon_R = 2, \mu_R = 3, \sigma_2 = 0$) toward region 1 ($\epsilon_{R1} = 4, \mu_{R1} = 2, \sigma_1 = 0$). If $\mathbf{B}_1 = (\mathbf{a}_x - 2\mathbf{a}_y + 3\mathbf{a}_z)\sin 300t$ T at point P in region 1 adjacent to the boundary, find the amplitude at P of: (*a*) $\mathbf{B}_{N1}$; (*b*) $\mathbf{B}_{t1}$; (*c*) $\mathbf{B}_{N2}$; (*d*) $\mathbf{B}_2$.

Ans. 2.00 T; 3.26 T; 2.00 T; 5.15 T

✔ **D10.6.** The surface $y = 0$ is a perfectly conducting plane, while the region $y > 0$ has $\epsilon_R = 5$, $\mu_R = 3$, and $\sigma = 0$. Let $\mathbf{E} = 20\cos(2\times 10^8 t - 2.58z)\mathbf{a}_y$ V/m for $y > 0$, and find at $t = 6$ ns; (*a*) ρ_S at $P(2, 0, 0.3)$; (*b*) $\mathbf{H}$ at P; (*c*) $\mathbf{K}$ at P.

Ans. 40.3 nC/m^2; $-62.3\mathbf{a}_x$ mA/m; $-62.3\mathbf{a}_z$ mA/m

10.5 THE RETARDED POTENTIALS

The time-varying potentials, usually called *retarded* potentials for a reason which we shall see shortly, find their greatest application in radiation problems in which the distribution of the source is known approximately. We should remember that the scalar electric potential V may be expressed in terms of a static charge distribution,

$$V = \int_{\text{vol}} \frac{\rho_v dv}{4\pi\epsilon R} \quad \text{(static)} \tag{45}$$

and the vector magnetic potential may be found from a current distribution which is constant with time,

$$\mathbf{A} = \int_{\text{vol}} \frac{\mu \mathbf{J}\, dv}{4\pi R} \quad \text{(dc)} \tag{46}$$

The differential equations satisfied by V,

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} \quad \text{(static)} \tag{47}$$

and $\mathbf{A}$,

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} \quad \text{(dc)} \tag{48}$$

may be regarded as the point forms of the integral equations (45) and (46), respectively.

Having found V and $\mathbf{A}$, the fundamental fields are then simply obtained by using the gradient,

$$\mathbf{E} = -\nabla V \quad \text{(static)} \tag{49}$$

or the curl,

$$\mathbf{B} = \nabla \times \mathbf{A} \quad \text{(dc)} \tag{50}$$

We now wish to define suitable time-varying potentials which are consistent with the above expressions when only static charges and direct currents are involved.

Equation (50) apparently is still consistent with Maxwell's equations. These equations state that $\nabla \cdot \mathbf{B} = 0$, and the divergence of (50) leads to the divergence of the curl which is identically zero. Let us therefore tentatively accept (50) as satisfactory for time-varying fields and turn our attention to (49).

The inadequacy of (49) is obvious, because application of the curl operation to each side and recognition of the curl of the gradient as being identically zero confront us with $\nabla \times \mathbf{E} = 0$. However, the point form of Faraday's law states that $\nabla \times \mathbf{E}$ is not generally zero, so let us try to effect an improvement by adding an unknown term to (49),

$$\mathbf{E} = -\nabla V + \mathbf{N}$$

taking the curl,

$$\nabla \times \mathbf{E} = 0 + \nabla \times \mathbf{N}$$

using the point form of Faraday's law,

$$\nabla \times \mathbf{N} = -\frac{\partial \mathbf{B}}{\partial t}$$

and using (50), giving us

$$\nabla \times \mathbf{N} = -\frac{\partial}{\partial t}(\nabla \times \mathbf{A})$$

or

$$\nabla \times \mathbf{N} = -\nabla \times \frac{\partial \mathbf{A}}{\partial t}$$

The simplest solution of this equation is

$$\mathbf{N} = -\frac{\partial \mathbf{A}}{\partial t}$$

and this leads to

$$\boxed{\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t}} \tag{51}$$

We still must check (50) and (51) by substituting them into the remaining two of Maxwell's equations:

$$\nabla \times \mathbf{H} = \mathbf{J} + \frac{\partial \mathbf{D}}{\partial t}$$

$$\nabla \cdot \mathbf{D} = \rho_v$$

Doing this, we obtain the more complicated expressions

$$\frac{1}{\mu}\nabla \times \nabla \times \mathbf{A} = \mathbf{J} + \epsilon\left(-\nabla\frac{\partial V}{\partial t} - \frac{\partial^2 \mathbf{A}}{\partial t^2}\right)$$

and

$$\epsilon\left(-\nabla \cdot \nabla V - \frac{\partial}{\partial t}\nabla \cdot \mathbf{A}\right) = \rho_v$$

or

$$\nabla(\nabla \cdot \mathbf{A}) - \nabla^2 \mathbf{A} = \mu \mathbf{J} - \mu\epsilon\left(\nabla\frac{\partial V}{\partial t} + \frac{\partial^2 \mathbf{A}}{\partial t^2}\right) \tag{52}$$

and

$$\nabla^2 V + \frac{\partial}{\partial t}(\nabla \cdot \mathbf{A}) = -\frac{\rho_v}{\epsilon} \tag{53}$$

There is no apparent inconsistency in (52) and (53). Under static or dc conditions $\nabla \cdot \mathbf{A} = 0$, and (52) and (53) reduce to (48) and (47), respectively. We shall therefore assume that the time-varying potentials may be defined in such a way that **B** and **E** may be obtained from them through (50) and (51). These latter two equations do not serve, however, to define **A** and V *completely*. They represent necessary, but not sufficient, conditions. Our initial assumption was merely that $B = \nabla \times \mathbf{A}$, and a vector cannot be defined by giving its curl alone. Suppose, for example, that we have a very simple vector potential field in which A_y and A_z are zero. Expansion of (50) leads to

$$B_x = 0$$

$$B_y = \frac{\partial A_x}{\partial z}$$

$$B_z = -\frac{\partial A_x}{\partial y}$$

and we see that no information is available about the manner in which A_x varies with x. This information could be found if we also knew the value of the divergence of **A**, for in our example

$$\nabla \cdot \mathbf{A} = \frac{\partial A_x}{\partial x}$$

Finally, we should note that our information about **A** is given only as partial derivatives and that a space-constant term might be added. In all physical problems in which the region of the solution extends to infinity, this constant term must be zero, for there can be no fields at infinity.

Generalizing from this simple example, we may say that a vector field is defined completely when both its curl and divergence are given and when its value is known at any one point (including infinity). We are therefore at liberty

to specify the divergence of **A**, and we do so with an eye on (52) and (53), seeking the simplest expressions. We define

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t} \tag{54}$$

and (52) and (53) become

$$\nabla^2 \mathbf{A} = -\mu \mathbf{J} + \mu\epsilon \frac{\partial^2 \mathbf{A}}{\partial t^2} \tag{55}$$

and

$$\nabla^2 V = -\frac{\rho_v}{\epsilon} + \mu\epsilon \frac{\partial^2 V}{\partial t^2} \tag{56}$$

These equations are related to the wave equation, which will be discussed in the following chapter. They show considerable symmetry, and we should be highly pleased with our definitions of V and **A**,

$$\mathbf{B} = \nabla \times \mathbf{A} \tag{50}$$

$$\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t} \tag{54}$$

$$\mathbf{E} = -\nabla V - \frac{\partial \mathbf{A}}{\partial t} \tag{51}$$

The integral equivalents of (45) and (46) for the time-varying potentials follow from the definitions (50), (51), and (54), but we shall merely present the final results and indicate their general nature. In the next chapter a study of the uniform plane wave will introduce the concept of *propagation*, in which any electromagnetic disturbance is found to travel at a velocity

$$v = \frac{1}{\sqrt{\mu\epsilon}}$$

through any homogeneous medium described by μ and ϵ. In the case of free space this velocity turns out to be velocity of light, approximately 3×10^8 m/s. It is logical, then, to suspect that the potential at any point is due not to the value of the charge density at some distant point at the same instant, but to its value at some previous time, because the effect propagates at a finite velocity. Thus (45) becomes

$$\boxed{V = \int_{\text{vol}} \frac{[\rho_v]}{4\pi\epsilon R} dv} \tag{57}$$

where $[\rho_v]$ indicates that every t appearing in the expression for ρ_v has been replaced by a *retarded* time,

$$t' = t - \frac{R}{v}$$

Thus, if the charge density throughout space were given by

$$\rho_v = e^{-r} \cos \omega t$$

then

$$[\rho_v] = e^{-r} \cos\left[\omega\left(t - \frac{R}{v}\right)\right]$$

where R is the distance between the differential element of charge being considered and the point at which the potential is to be determined.

The retarded vector magnetic potential is given by

$$\boxed{\mathbf{A} = \int_{\mathrm{vol}} \frac{\mu[\mathbf{J}]}{4\pi R} dv} \tag{58}$$

The use of a retarded time has resulted in the time-varying potentials being given the name of retarded potentials. In Chap. 13 we shall apply (58) to the simple situation of a differential current element in which I is a sinusoidal function of time. Other simple applications of (58) are considered in several problems at the end of this chapter.

We may summarize the use of the potentials by stating that a knowledge of the distribution of ρ_v and $\mathbf{J}$ throughout space theoretically enables us to determine V and $\mathbf{A}$ from (57) and (58). The electric and magnetic fields are then obtained by applying (50) and (51). If the charge and current distributions are unknown, or reasonable approximations cannot be made for them, these potentials usually offer no easier path toward the solution than does the direct application of Maxwell's equations.

✔ **D10.7.** A point charge of $4 \cos 10^8 \pi t\, \mu\mathrm{C}$ is located at $P_+(0, 0, 1.5)$, while $-4 \cos 10^8 \pi t\, \mu\mathrm{C}$ is at $P_-(0, 0, -1.5)$, both in free space. Find V at $P(r = 450, \theta, \phi = 0)$ at $t = 15$ ns for $\theta =$: (*a*) 0°; (*b*) 90°; (*c*) 45°.

Ans. 159.8 V; 0; 107.1 V

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6. Owen, G. E.: "Electromagnetic Theory," Allyn and Bacon, Inc., Boston, 1963. Faraday's law is discussed in terms of the frame of reference in chap. 8.
7. Panofsky, W. K. H. and M. Phillips: "Classical Electricity and Magnetism," 2d ed., Addison-Wesley Publishing Company, Inc., Reading, Mass., 1962. Relativity is treated at a moderately advanced level in chap. 15.

PROBLEMS

10.1 In Fig. 10.4, let $B = 0.2\cos 120\pi t$ T, and assume that the conductor joining the two ends of the resistor is perfect. It may be assumed that the magnetic field produced by $I(t)$ is negligible. Find: (*a*) $V_{ab}(t)$; (*b*) $I(t)$.

10.2 Given the time-varying magnetic field $\mathbf{B} = (0.5\mathbf{a}_x + 0.6\mathbf{a}_y - 0.3\mathbf{a}_z)\cos 5000t$ T and a square filamentary loop with its corners at (2,3,0), (2,−3,0), (−2,3,0), and (−2,−3,0), find the time-varying current flowing in the general $\mathbf{a}_\phi$ direction if the total loop resistance is 400 kΩ.

10.3 Given $\mathbf{H} = 300\mathbf{a}_z\cos(3\times 10^8 t - y)$ A/m in free space, find the emf developed in the general $\mathbf{a}_\phi$ direction about the closed path having corners at: (*a*) (0,0,0), (1,0,0), (1,1,0), and (0,1,0); (*b*) (0,0,0) $(2\pi,0,0)$, $(2\pi,2\pi,0)$, $(0, 2\pi,0)$.

10.4 Conductor surfaces are located at $\rho = 1$ cm and $\rho = 2$ cm in free space. The volume 1 cm $< \rho < 2$ cm contains the fields $H_\phi =$

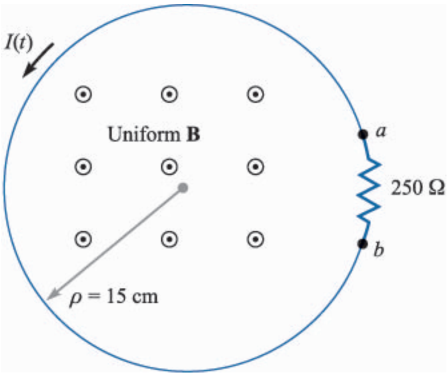

FIGURE 10.4
See Prob. 10.1.

$\dfrac{2}{\rho}\cos(6 \times 10^8\pi t - 2\pi z)$ A/m and $E_\rho = \dfrac{240\pi}{\rho}\cos(6 \times 10^8\pi t - 2\pi z)$ V/m. (*a*) Show that these two fields satisfy Eq. (6), Sec. 10.1. (*b*) Evaluate both integrals in Eq. (4) for the planar surface defined by $\phi = 0$, 1 cm $< \rho <$ 2 cm, $z = 0.1$, and its perimeter, and show that the same results are obtained.

10.5 The location of the sliding bar in Figure 10.5 is given by $x = 5t + 2t^3$, and the separation of the two rails is 20 cm. Let $\mathbf{B} = 0.8x^2\mathbf{a}_z$ T. Find the voltmeter reading at: (*a*) $t = 0.4$ s; (*b*) $x = 0.6$ m.

10.6 A perfectly conducting filament containing a small 500-Ω resistor is formed into a square, as illustrated by Fig. 10.6. Find $I(t)$ if $\mathbf{B} =$: (*a*) $0.3\cos(120\pi t - 30^\circ)\mathbf{a}_z$ T; (*b*) $0.4\cos[\pi(ct - y)]\mathbf{a}_z\ \mu$T, where $c = 3 \times 10^8$ m/s.

10.7 The rails in Fig. 10.7 each have a resistance of 2.2 Ω/m. The bar moves to the right at a constant speed of 9 m/s in a uniform magnetic field of 0.8 T. Find $I(t)$, $0 < t < 1$ s, if the bar is at $x = 2$ m at $t = 0$ and: (*a*) a 0.3-Ω resistor is present across the left end with the right end open-circuited; (*b*) a 0.3-Ω resistor is present across each end.

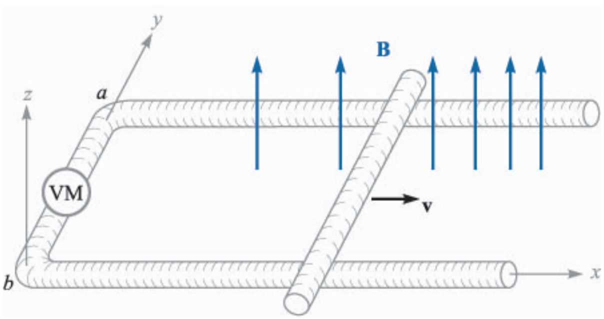

FIGURE 10.5
See Prob. 10.5.

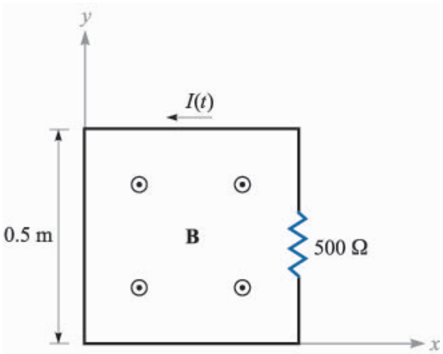

FIGURE 10.6
See Prob. 10.6.

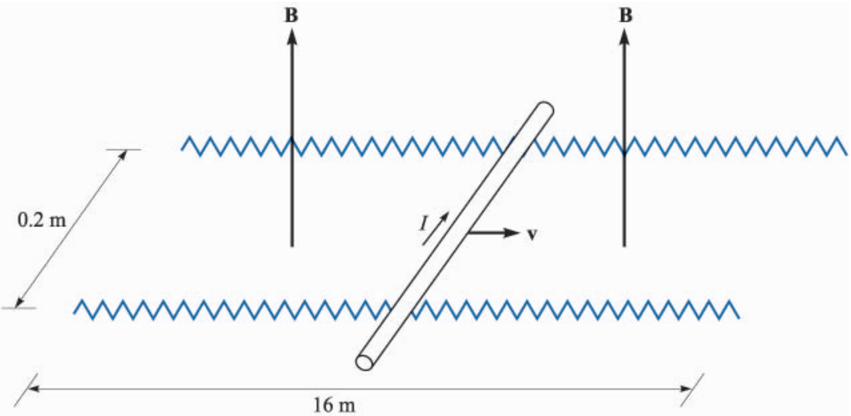

FIGURE 10.7
See Prob. 10.7.

10.8 Fig. 10.1 is modified to show that the rail separation is larger when y is larger. Specifically, let the separation $d = 0.2 + 0.02y$. Given a uniform velocity $v_y = 8$ m/s and a uniform magnetic flux density $B_z = 1.1$ T, find V_{12} as a function of time if the bar is located at $y = 0$ at $t = 0$.

10.9 A square filamentary loop of wire is 25 cm on a side and has a resistance of 125 Ω per meter length. The loop lies in the $z = 0$ plane with its corners at (0,0,0), (0.25,0,0), (0.25,0.25,0), and (0,0.25,0) at $t = 0$. The loop is moving with a velocity $v_y = 50$ m/s in the field $B_z = 8\cos(1.5 \times 10^8 t - 0.5x)\,\mu$T. Develop a function of time which expresses the ohmic power being delivered to the loop.

10.10 (*a*) Show that the ratio of the amplitudes of the conduction current density and the displacement current density is $\sigma/\omega\epsilon$ for the applied field $E = E_m \cos \omega t$. Assume $\mu = \mu_0$. (*b*) What is the amplitude ratio if the applied field is $E = E_m e^{-t/\tau}$, where τ is real?

10.11 Let the internal dimensions of a coaxial capacitor be $a = 1.2$ cm, $b = 4$ cm, and $l = 40$ cm. The homogeneous material inside the capacitor has the parameters $\epsilon = 10^{-11}$ F/m, $\mu = 10^{-5}$ H/m, and $\sigma = 10^{-5}$ S/m. If the electric field intensity is $\mathbf{E} = (10^6/\rho)\cos 10^5 \mathbf{a}_\rho$ V/m, find: (*a*) **J**; (*b*) the total conduction current I_c through the capacitor; (*c*) the total displacement current I_d through the capacitor; (*d*) the ratio of the amplitude of I_d to that of I_c, the quality factor of the capacitor.

10.12 Given a coaxial transmission line with $\dfrac{b}{a} = e^{2.5}$, $\mu_R = \epsilon_R = 1$, and an electric field intensity $\mathbf{E} = \dfrac{200}{\rho}\cos(10^9 t - 3.336z)\mathbf{a}_\rho$ V/m, find: (*a*) V_{ab}, the voltage between the conductors, if it is known that the electrostatic relationship $\mathbf{E} = -\nabla V$ is valid; (*b*) the displacement current density.

10.13 Consider the region defined by $|x|$, $|y|$, and $|z| < 1$. Let $\epsilon_R = 5$, $\mu_R = 4$, and $\sigma = 0$. If $J_d = 20\cos(1.5 \times 10^8 t - bx)\mathbf{a}_y\ \mu\text{A/m}^2$: (*a*) find **D** and **E**; (*b*) use the point form of Faraday's law and an integration with respect to time to find **B** and **H**; (*c*) use $\nabla \times \mathbf{H} = \mathbf{J}_d + \mathbf{J}$ to find $\mathbf{J}_d$. (*d*) What is the numerical value of b?

10.14 A voltage source $V_0 \sin \omega t$ is connected between two concentric conducting spheres, $r = a$ and $r = b$, $b > a$, where the region between them is a material for which $\epsilon = \epsilon_R \epsilon_0$, $\mu = \mu_0$, and $\sigma = 0$. Find the total displacement current through the dielectric and compare it with the source current as determined from the capacitance (Sec. 5.10) and circuit-analysis methods.

10.15 Let $\mu = 3 \times 10^{-5}$ H/m, $\epsilon = 1.2 \times 10^{-10}$ F/m, and $\sigma = 0$ everywhere. If $\mathbf{H} = 2\cos(10^{10} t - \beta x)\mathbf{a}_z$ A/m, use Maxwell's equations to obtain expressions for **B**, **D**, **E**, and β.

10.16 (*a*) A certain material has $\sigma = 0$ and $\epsilon_R = 1$. If $\mathbf{H} = 4\sin(10^6 t - 0.01z)\mathbf{a}_y$ A/m, make use of Maxwell's equations to find μ_R. (*b*) Find $\mathbf{E}(z, t)$.

10.17 The electric field intensity in the region $0 < x < 5$, $0 < y < \pi/12$, $0 < z < 0.06$ m in free space is given by $\mathbf{E} = C\sin 12y \sin az \cos 2 \times 10^{10} t\mathbf{a}_x$ V/m. Beginning with the $\nabla \times \mathbf{E}$ relationship, use Maxwell's equations to find a numerical value for a, if it is known that a is greater than zero.

10.18 The parallel-plate transmission line shown in Fig. 10.8 has dimensions $b = 4$ cm and $d = 8$ mm, while the medium between the plates is characterized by $\mu_R = 1$, $\epsilon_R = 20$, and $\sigma = 0$. Neglect fields outside the dielectric. Given the field $\mathbf{H} = 5\cos(10^9 t - \beta z)\mathbf{a}_y$ A/m, use Maxwell's equations to help find: (*a*) β, if $\beta > 0$; (*b*) the displacement current density at $z = 0$; (*c*) the total displacement current crossing the surface $x = 0.5d$, $0 < y < b$, $0 < z < 0.1$ m in the $\mathbf{a}_x$ direction.

10.19 In the first section of this chapter, Faraday's law was used to show that the field $\mathbf{E} = -\frac{1}{2}kB_0 e^{kt}\mathbf{a}_\phi$ results from the changing magnetic field $\mathbf{B} = B_0 e^{kt}\mathbf{a}_z$. (*a*) Show that these fields do not satisfy Maxwell's other

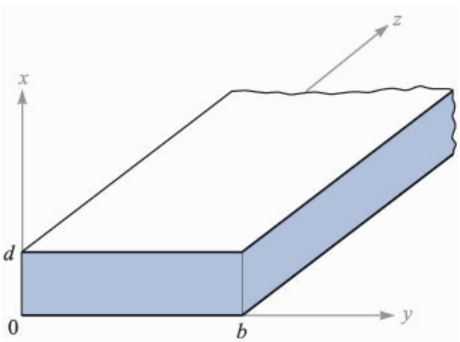

FIGURE 10.8
See Prob. 10.18.

curl equation. (*b*) If we let $B_0 = 1$ T and $k = 10^6\ s^{-1}$, we are establishing a fairly large magnetic flux density in 1 μs. Use the $\nabla \times \mathbf{H}$ equation to show that the rate at which B_z should (but does not) change with ρ is only about 5×10^{-6} T per meter in free space at $t = 0$.

10.20 Point $C(-0.1, -0.2, 0.3)$ lies on the surface of a perfect conductor. The electric field intensity at C is $(500\mathbf{a}_x - 300\mathbf{a}_y + 600\mathbf{a}_z)\cos 10^7 t$ V/m, and the medium surrounding the conductor is characterized by $\mu_R = 5$, $\epsilon_R = 10$, and $\sigma = 0$. (*a*) Find a unit vector normal to the conductor surface at C, if the origin lies within the conductor. (*b*) Find the surface charge density at C.

10.21 The surfaces $\rho = 3$ and 10 mm, and $z = 0$ and 25 cm are perfect conductors. The region enclosed by these surfaces has $\mu = 2.5 \times 10^{-6}$ H/m, $\epsilon = 4 \times 10^{-11}$ F/m, and $\sigma = 0$. Let $\mathbf{H} = (2/\rho)\cos 10\pi z \cos \omega t \mathbf{a}_\phi$ A/m. Make use of Maxwell's equations to find: (*a*) ω; (*b*) $\mathbf{E}$.

10.22 In free space, where $\epsilon = \epsilon_0$, $\mu = \mu_0$, $\sigma = 0$, $\mathbf{J} = 0$, and $\rho_v = 0$, assume a cartesian coordinate system in which $\mathbf{E}$ and $\mathbf{H}$ are both functions only of z and t. (*a*) If $\mathbf{E} = E_y \mathbf{a}_y$ and $\mathbf{H} = H_x \mathbf{a}_x$, begin with Maxwell's equations and determine the second-order partial differential equation that E_y must satisfy. (*b*) Show that $E_y = 5(300t + bz)^2$ is a solution of that equation for a particular value of b, and find that value.

10.23 In region 1, $z < 0$, $\epsilon_1 = 2 \times 10^{-11}$ F/m, $\mu_1 = 2 \times 10^{-6}$ H/m, and $\sigma_1 = 4 \times 10^{-3}$ S/m; in region 2, $z > 0$, $\epsilon_2 = \epsilon_1/2$, $\mu_2 = 2\mu_1$, and $\sigma_2 = \sigma_1/4$. It is known that $\mathbf{E}_1 = (30\mathbf{a}_x + 20\mathbf{a}_y + 10\mathbf{a}_z)\cos 10^9 t$ V/m at $P(0,0,0^-)$. (*a*) Find $\mathbf{E}_{N1}$, $\mathbf{E}_{t1}$, $\mathbf{D}_{N1}$, and $\mathbf{D}_{t1}$ at P_1. (*b*) Find $\mathbf{J}_{N1}$ and $\mathbf{J}_{t1}$ at P_1. (*c*) Find $\mathbf{E}_{t2}$, $\mathbf{D}_{t2}$, and $\mathbf{J}_{t2}$ at $P_2(0, 0, 0^+)$. (*d*) (Harder) Use the continuity equation to help show that $J_{N1} - J_{N2} = \partial D_{N2}/\partial t$, and then determine $\mathbf{D}_{N2}$, $\mathbf{J}_{N2}$, and $\mathbf{E}_{N2}$.

10.24 Given the fields $V = 80z \cos x \cos 3 \times 10^8 t$ kV and $\mathbf{A} = 26.7z \sin x \sin 3 \times 10^8 t \mathbf{a}_x$ mWb/m in free space, find $\mathbf{E}$ and $\mathbf{H}$.

10.25 In a region where $\mu_R = \epsilon_R = 1$ and $\sigma = 0$, the retarded potentials are given by $V = x(z - ct)$ V and $\mathbf{A} = x\left(\frac{z}{c} - t\right)\mathbf{a}_z$ Wb/m, where $c = 1\sqrt{\mu_0\epsilon_0}$. (*a*) Show that $\nabla \cdot \mathbf{A} = -\mu\epsilon \frac{\partial V}{\partial t}$. (*b*) Find $\mathbf{B}$, $\mathbf{H}$, $\mathbf{E}$, and $\mathbf{D}$. (*c*) Show that these results satisfy Maxwell's equations if $\mathbf{J}$ and ρ_v are zero.

10.26 Let the current $I = 80t$ A be present in the $\mathbf{a}_z$ direction on the z axis in free space within the interval $-0.1 < z < 0.1$ m. (*a*) Find A_z at $P(0, 2, 0)$ and: (*b*) sketch A_z versus t over the time interval $-0.1 < t < 0.1$ μs.